Statistical Physics of Simple and Complex Fluids

Soft Condensed Matter Theory

HOMEWORK #3

Interfaces and membranes: thermal fluctuations and Helfrich forces

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1 Spatial correlation function

Let $z(\mathbf{r})$ be the height of a fluctuating interface at position $\mathbf{r} = (x, y) \in [0, L] \times [0, L]$. We want to characterize the spatial correlations of the interface height. We define the *spatial correlation function* as:

$$g(\mathbf{r}) = \langle z(\mathbf{r}) z(\mathbf{0}) \rangle. \tag{1}$$

1. Show that
$$g(\mathbf{r}) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \tilde{z}(\mathbf{k}) \tilde{z}^*(\mathbf{k}') \rangle e^{\imath \mathbf{k} \cdot \mathbf{r}}. \tag{2}$$
What are the possible values for $\mathbf{k}$ (and $\mathbf{k}'$), using periodic boundary conditions ?

2. Show that
$$\langle \tilde{z}(\mathbf{k}) \tilde{z}^*(\mathbf{k}') \rangle = \frac{k_B T}{\gamma L^2 (k^2 + l_c^{-2})} \delta_{\mathbf{k},\mathbf{k}'}, \tag{3}$$
where l_c is the capillary length, and $\delta_{\mathbf{k},\mathbf{k}'}$ the Kronecker symbol.

3. How are modified Eqs (2) and (3) if $\mathbf{k}$ and $\mathbf{k}'$ are treated as continuous variables ?

4. Using this continuous approximation, finally show that
$$g(\mathbf{r}) = \frac{k_B T}{2\pi\gamma} K_0 \left(\frac{r}{l_c} \right), \tag{4}$$
where $K_0(x) = \int_0^\infty \int_0^{2\pi} \frac{e^{\imath k r \cos\theta}}{k^2 + l_c^{-2}} k \mathrm{d}\theta \mathrm{d}k$ is the modified Bessel function of the second kind at zeroth order.

5. It can be shown that, when $x \gg 1$:
$$K_0(x) \simeq \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}.$$
Make a plot of $g(\mathbf{r})$. What is the characteristic correlation length ?

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2 Entropic interactions between a membrane and walls (Helfrich forces)

When a fluctuating membrane is close to a wall or another membrane, it experiences an entropically-driven repulsive force. The aim of this exercise is to determine the range of this interaction. To simplify, we consider the case of a membrane which fluctuates between two parallel walls.

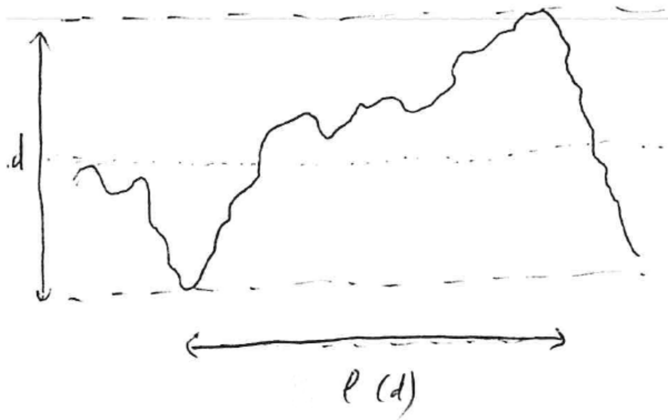

Figure 1: Entropic interactions between a fluctuating membrane and walls.

2.1 Qualitative approach #1: kinetic pressure exerted on the membrane

A first interpretation of this force is that it results from collisions between the membrane a each wall. We then look for the number of collisions per unit surface area of the system.

1. We define the length $l(d)$ such that a membrane with length $l(d)$ has a height standard deviation equal to d, the distance between the two walls (see Fig. 1). One reminds that for a membrane with length L:

$$\langle h^2 \rangle = \frac{k_\mathrm{B}T}{8\pi^3 \kappa_b} L^2,$$

where κ_b is the bending rigidity of the membrane. Deduce the typical distance $l(d)$ between two contact points.

2. Show that the kinetic pressure of the "ideal gas" of contact points is:

$$P = \frac{k_\mathrm{B}T}{2l(d)^2 d}.$$

3. The Free energy (per unit area) due to the wall-membrane interaction, ΔF, is equal to the work required to bring the parallel walls from distance $d' = \infty$ to $d' = d$. Deduce that:

$$\Delta F \sim \frac{(k_\mathrm{B}T)^2}{\kappa_b d^2}.$$

Comment on the range (short or long) of this interaction.

4. Attractive Van der Waals interactions between two parallel planes scales as $-A/d^2$, where A is a constant independant of temperature. Comment on a possible detachment transition with temperature.

2.2 Qualitative approach #2: extinction of the long wawelength modes

According to the equipartition theorem, in absence of walls the contribution to the mean energy of each Fourier mode of the height profile is $k_\mathrm{B}T$. In presence of walls, each contact point freezes one Fourier mode (starting from longer wavelengths), such that $\Delta F = \Gamma k_\mathrm{B}T$, where Γ is the number of contact points per unit surface area. Deduce the expression of ΔF as a function of d.

2.3 Quantitative approach: harmonic interaction

In order to calculate more precisely the interaction between the walls, we replace the hard-core (steric) interactions with the walls by a an harmonic potential:

$$V(h) = mh^2/2,$$

where $h(x, y)$ is the height of the membrane with respect to the midplane betwwen the two walls, and m is the stiffness of the potential whose value will be chosen latter.

1. Justify that the fluctuation spectrum of a tensionless membrane submitted to the harmonic potential is:

$$\left\langle |\tilde{h}(\mathbf{q})|^2 \right\rangle = \frac{k_\mathrm{B}T}{L^2(\kappa_b q^4 + m)},$$

   where $\mathbf{q}$ is a two-dimensional wave vector and L the lateral length of the considered membrane patch.

2. Using the identity

$$\int_0^\infty \frac{q}{\kappa_b q^4 + m} dq = \frac{\pi}{4\sqrt{\kappa_b m}},$$

   give the expression of $\left\langle h^2 \right\rangle$.

3. Justify the following choice for the membrane stiffness:

$$m = \frac{(k_\mathrm{B}T)^2}{\kappa_b d^4}.$$

4. Calculate the partition function of the membrane. One reminds that:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2/2} = \sqrt{\frac{2\pi}{\alpha}}.$$

   Express the free energy as an integral over wave vector values.

5. Claculate the pressure $P(d)$ as a derivative of the free energy per unit surface area. Check that:

$$P(d) \sim \frac{(k_\mathrm{B}T)^2}{\kappa_b d^3}.$$