

# STATISTICAL PHYSICS OF SIMPLE AND COMPLEX FLUIDS

## SOFT CONDENSED MATTER THEORY

### HOMEWORK #3

#### Interfaces and membranes: thermal fluctuations and Helfrich forces

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## 1 Spatial correlation function

Let  $z(\mathbf{r})$  be the height of a fluctuating interface at position  $\mathbf{r} = (x, y) \in [0, L] \times [0, L]$ . We want to characterize the spatial correlations of the interface height. We define the *spatial correlation function* as:

$$g(\mathbf{r}) = \langle z(\mathbf{r})z(\mathbf{0}) \rangle. \quad (1)$$

1. Show that

$$g(\mathbf{r}) = \sum_{\mathbf{k}} \sum_{\mathbf{k}'} \langle \tilde{z}(\mathbf{k})\tilde{z}^*(\mathbf{k}') \rangle e^{i\mathbf{k}\cdot\mathbf{r}}. \quad (2)$$

What are the possible values for  $\mathbf{k}$  (and  $\mathbf{k}'$ ), using periodic boundary conditions ?

2. Show that

$$\langle \tilde{z}(\mathbf{k})\tilde{z}^*(\mathbf{k}') \rangle = \frac{k_B T}{\gamma L^2 (k^2 + l_c^{-2})} \delta_{\mathbf{k}, \mathbf{k}'}, \quad (3)$$

where  $l_c$  is the capillary length, and  $\delta_{\mathbf{k}, \mathbf{k}'}$  the Kronecker symbol.

3. How are modified Eqs (2) and (3) if  $\mathbf{k}$  and  $\mathbf{k}'$  are treated as continuous variables ?  
 4. Using this continuous approximation, finally show that

$$g(\mathbf{r}) = \frac{k_B T}{2\pi\gamma} K_0\left(\frac{r}{l_c}\right), \quad (4)$$

where  $K_0(x) = \int_0^\infty \int_0^{2\pi} \frac{e^{+ikr \cos \theta}}{k^2 + l_c^{-2}} k d\theta dk$  is the modified Bessel function of the second kind at zeroth order.

5. It can be shown that, when  $x \gg 1$ :

$$K_0(x) \simeq \sqrt{\frac{\pi}{2}} \frac{e^{-x}}{\sqrt{x}}.$$

Make a plot of  $g(\mathbf{r})$ . What is the characteristic correlation length ?

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## 2 Entropic interactions between a membrane and walls (Helfrich forces)

When a fluctuating membrane is close to a wall or another membrane, it experiences an entropically-driven repulsive force. The aim of this exercise is to determine the range of this interaction. To simplify, we consider the case of a membrane which fluctuates between two parallel walls.

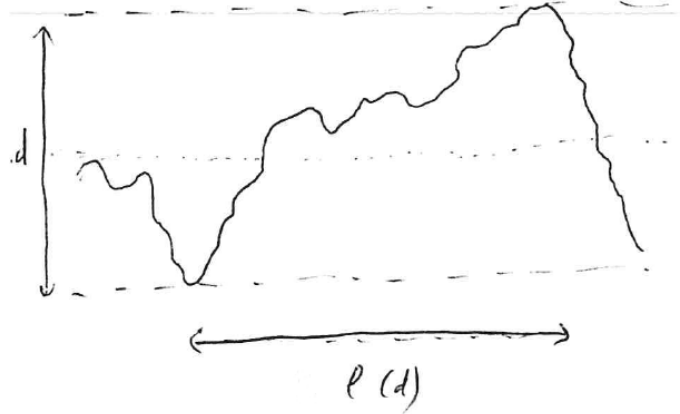


Figure 1: Entropic interactions between a fluctuating membrane and walls.

## 2.1 Qualitative approach #1: kinetic pressure exerted on the membrane

A first interpretation of this force is that it results from collisions between the membrane and each wall. We then look for the number of collisions per unit surface area of the system.

1. We define the length  $l(d)$  such that a membrane with length  $l(d)$  has a height standard deviation equal to  $d$ , the distance between the two walls (see Fig. 1). One reminds that for a membrane with length  $L$ :

$$\langle h^2 \rangle = \frac{k_B T}{8\pi^3 \kappa_b} L^2,$$

where  $\kappa_b$  is the bending rigidity of the membrane. Deduce the typical distance  $l(d)$  between two contact points.

2. Show that the kinetic pressure of the “ideal gas” of contact points is:

$$P = \frac{k_B T}{2l(d)^2 d}.$$

3. The Free energy (per unit area) due to the wall-membrane interaction,  $\Delta F$ , is equal to the work required to bring the parallel walls from distance  $d' = \infty$  to  $d' = d$ . Deduce that:

$$\Delta F \sim \frac{(k_B T)^2}{\kappa_b d^2}.$$

Comment on the range (short or long) of this interaction.

4. Attractive Van der Waals interactions between two parallel planes scales as  $-A/d^2$ , where  $A$  is a constant independent of temperature. Comment on a possible detachment transition with temperature.

## 2.2 Qualitative approach #2: extinction of the long wavelength modes

According to the equipartition theorem, in absence of walls the contribution to the mean energy of each Fourier mode of the height profile is  $k_B T$ . In presence of walls, each contact point freezes one Fourier mode (starting from longer wavelengths), such that  $\Delta F = \Gamma k_B T$ , where  $\Gamma$  is the number of contact points per unit surface area. Deduce the expression of  $\Delta F$  as a function of  $d$ .

### 2.3 Quantitative approach: harmonic interaction

In order to calculate more precisely the interaction between the walls, we replace the hard-core (steric) interactions with the walls by a harmonic potential:

$$V(h) = mh^2/2,$$

where  $h(x, y)$  is the height of the membrane with respect to the midplane between the two walls, and  $m$  is the stiffness of the potential whose value will be chosen later.

1. Justify that the fluctuation spectrum of a tensionless membrane submitted to the harmonic potential is:

$$\langle |\tilde{h}(\mathbf{q})|^2 \rangle = \frac{k_B T}{L^2(\kappa_b q^4 + m)},$$

where  $\mathbf{q}$  is a two-dimensional wave vector and  $L$  the lateral length of the considered membrane patch.

2. Using the identity

$$\int_0^\infty \frac{q}{\kappa_b q^4 + m} dq = \frac{\pi}{4\sqrt{\kappa_b m}},$$

give the expression of  $\langle h^2 \rangle$ .

3. Justify the following choice for the membrane stiffness:

$$m = \frac{(k_B T)^2}{\kappa_b d^4}.$$

4. Calculate the partition function of the membrane. One reminds that:

$$\int_{-\infty}^{+\infty} e^{-\alpha x^2/2} = \sqrt{\frac{2\pi}{\alpha}}.$$

Express the free energy as an integral over wave vector values.

5. Calculate the pressure  $P(d)$  as a derivative of the free energy per unit surface area. Check that:

$$P(d) \sim \frac{(k_B T)^2}{\kappa_b d^3}.$$