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Maximum electrical conductivity of a network of uniform wires: the Lemlich law as an upper bound

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The Lemlich law provides a simple estimate of the relative conductivity of a three-dimensional foam, as one third of its liquid fraction. This is based on an expression for the conductivity of a network of uniform wires (conducting lines). We show this to be an exact upper bound for the conductivity, orientationally averaged in the case of anisotropic systems. We discuss the dependance of conductivity on the geometry of the network structure and establish two necessary and sufficient conditions to maximize the conductivity. We note the connection between this problem and that of line-length minimization and also that between anisotropic conductivity and stress for a two-dimensional foam. These results are illustrated by various numerical simulations of network conductivities. The theorems presented in this paper may also be applied to the thermal conductivity and the permeability of a network.

Keywords: network; upper bound; conductivity; maximum; foam

1. Introduction

Recent interest in the conductivity of foams has led naturally to the theory of networks of uniform wires. Various approximations, detailed later in this paper, reduce the idealized model of the foam to this form, particularly in the dry limit, in which the liquid fraction, defined as the volume of liquid per unit volume of foam, becomes small. In this way Lemlich (1978) produced a classic estimate of effective foam conductivity σ, which is

$$\frac{\sigma}{\sigma_{\text{liq}}} = \tfrac{1}{3}\phi_{\text{l}}. \tag{1.1}$$

Here σ_{liq} is the conductivity of the liquid from which the foam is composed, and ϕ_{l} is the liquid fraction. Numerous experiments (Chang & Lemlich 1980; Phelan *et al.* 1996; Weaire & Hutzler 1999) have confirmed the approximate validity of the Lemlich formula, in the dry limit. Extensions of this model at higher liquid fractions were formulated by Agnihotri & Lemlich (1981) by including the contribution of the lamellae, and by Phelan *et al.* (1996) by including the contribution of the nodes to the distribution of liquid and the effective conductivity of the foam. Since the cross-section of a node is higher than the cross-section of a channel, this correction increases the effective conductivity.

At the heart of Lemlich's derivation is the following estimate for the conductivity of a network composed of uniform wires (i.e. conducting lines) of resistance per unit length r:

$$\sigma \approx \frac{1}{3}\frac{\hat{l}}{r} \tag{1.2}$$

This includes the parameter $\hat{l}$, representing the line length of the network per unit volume. In general, equation (1.2) has been adduced without any explanation of its precise significance or conditions for being exact. It is the primary purpose of this paper to supply such an analysis. We shall see that this equation may be written as a *general exact upper bound* of the conductivity for such a network,

$$\sigma \leqslant \frac{1}{3}\frac{\hat{l}}{r}, \tag{1.3}$$

and that equality is attained in various cases. We shall note several of these and perform some numerical evaluations for cases in which the equality is not exact.

The inequality (1.3) applies on a network in a three-dimensional (3D) space, and has to be replaced in the case of a network in a two-dimensional space by

$$\sigma \leqslant \frac{1}{2}\frac{\hat{l}}{r}. \tag{1.4}$$

In what follows we shall concentrate on the case of a 3D network.

Various conditions are implicit in the above discussion. The homogeneity of the network on some large scale is assumed, to enable σ to be defined. Also it is assumed that the network has an isotropic conductivity. For ordered networks this is assured by cubic symmetry (Dubrovin *et al.* 1992), and it is often assumed to be the case for a typical disordered foam (see, however, the discussion of stress below). Alternatively, inequality (1.3) may be taken to apply to the orientationally averaged conductivity $\langle\sigma\rangle$ (see Appendix A), whenever the conductivity is not isotropic.

Having identified the relevant formula (1.3) as an exact upper bound and seen that it is also an excellent estimate in many cases, we shall discuss how it is used in the case of a foam, to arrive at Lemlich's formula, which may accordingly be viewed as an approximate upper bound, in the dry limit.

2. Derivation of upper bound for network conductivity

We have in mind a network made of straight wires in three dimensions, as illustrated by figure 1. Its conductivity may be bounded by use of the general method of 'cut-and-short', due to Lord Rayleigh (Jeans 1925; Maxwell 1891; Rayleigh 1899). This method is based on the following theorem (sometimes called the monotonicity law).

Theorem 2.1. *If any of the resistances of a circuit are increased, the effective resistance between any two points can only increase. If they are decreased, the effective resistance can only decrease.*

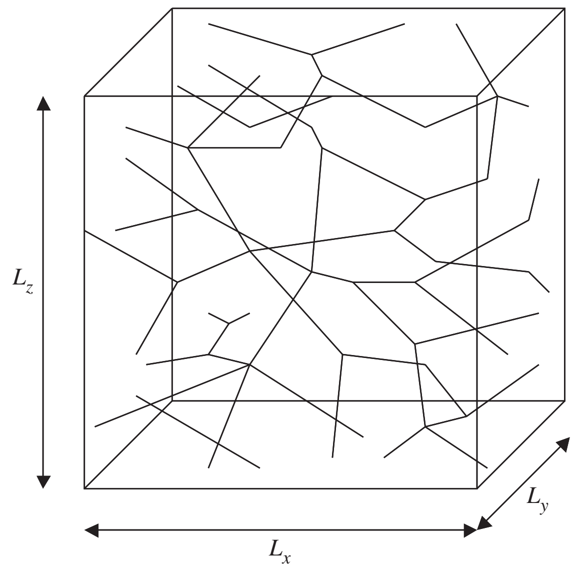

Figure 1. A 3D network composed of uniform wires of equal resistance per unit length r. L_x, L_y, L_z are the macroscopic dimensions on which the effective conductivity tensor of the network is defined.

This theorem is a consequence of the minimum Joule heating theorem (also called Thomson's principle). In terms appropriate to our case, this states that, for a given steady total current, the distribution of currents corresponds to a minimum of dissipation.

We reproduce here the proof of the monotonicity law (Jeans 1925; Maxwell 1891).

Proof. For a given circuit made of resistances r_α carrying by currents i_α, the total dissipation is equal to

$$\sum_\alpha r_\alpha i_\alpha^2 = RI^2,$$

where R is the equivalent resistance and I the total current. Let us suppose that each resistance is decreased from r_α to r'_α. If we imagine that currents remain unaltered, the new total dissipation would be $\sum_\alpha r'_\alpha i_\alpha^2$, which is less than the preceding one. But since the currents are not the real ones, the minimum of dissipation is not reached, and if we now allow currents to distribute themselves in the correct way, there will be a further decay of the dissipation. The new dissipative energy is

$$\sum_\alpha r'_\alpha i'^2_\alpha = R' I^2,$$

where the i'_α are the natural currents and R' is the new equivalent resistance. It follows that $R' \leqslant R$. We can similarly prove that the increase in resistances in the network will produce an increase in the total resistance. ■

Note that Cohn's theorem and Pezari's theorem can give quantitatively the variation of the effective resistance of a network with the variations of resistances (Penfield *et al.* 1970).

The idea of Rayleigh was that shorting a circuit between two points is equivalent to decreasing to zero the resistance between these two points, and consequently the effective resistance of the circuit only decreases (or in special cases stays stationary). Cutting the resistance between two points is equivalent to increasing the resistance to infinity and cannot decrease the effective resistance of the network.

We can use the Rayleigh cut-and-short method as follows. The network is modified by shorting it with thin parallel sheets of infinite conductivity, perpendicular to the direction x of the applied potential difference, and infinitesimally separated from each other by Δx. The resistance $\Delta R(x)$ of the network slice at position x is attributable to the wires which cross it, acting in parallel, and can be easily evaluated (see Appendix B). The equivalent resistance of the shorted network along the x-direction is equal to the sum of the elementary resistive slices, and we can show that this resistance (or equivalently its conductivity) is bounded as

$$\sigma_x^{(s)} \leqslant \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2 \alpha_{ij}}{r}, \tag{2.1}$$

where V is the volume over which the effective conductivity tensor of the network is defined, l_{ij} is the length of the wire (i, j) and α_{ij} is the angle between this wire and the x-axis (for details see Appendix B). Indices (i) and (j) denote junctions, and the sum is carried on all the wires that compose the network. From the monotonicity law, it follows that the conductivity in the x-direction of the original network is always less than the shorted network:

$$\sigma_x \leqslant \sigma_x^{(s)} \leqslant \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2 \alpha_{ij}}{r}. \tag{2.2}$$

By using the same procedure in directions y and z, respectively, we obtain upper bounds for conductivity along these two directions:

$$\sigma_y \leqslant \sigma_y^{(s)} \leqslant \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2 \beta_{ij}}{r}, \tag{2.3}$$

$$\sigma_z \leqslant \sigma_z^{(s)} \leqslant \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2 \gamma_{ij}}{r}, \tag{2.4}$$

where β_{ij} and γ_{ij} are the angles between the channel (i, j) and the two axes y and z (defined such that $\cos \beta_{ij} \geqslant 0$ and $\cos \gamma_{ij} \geqslant 0$). For any wire (i, j), the three director cosines are related by

$$\cos^2 \alpha_{ij} + \cos^2 \beta_{ij} + \cos^2 \gamma_{ij} = 1, \tag{2.5}$$

and so the averaged conductivity $\langle \sigma \rangle$ (defined in Appendix A) is bounded by

$$\langle \sigma \rangle \leqslant \frac{1}{3rV} \sum_{(i,j)} l_{ij}. \tag{2.6}$$

The line length per unit volume of the network is equal to

$$\hat{l} = \frac{1}{V} \sum_{(i,j)} l_{ij};$$

thus, inequality (1.3) is proved for the case of a network made of straight wires. The inequality also holds for a network made of curved wires, since curving one wire can only increase the line length on one hand, and decrease the average conductivity of the network (from the monotonicity law) on the other hand.

3. Conditions for exact equality

We have just shown that the averaged conductivity of a network is always smaller than $\hat{l}/3r$, where $\hat{l}$ is its line length per unit volume, or equivalently the *normalized conductivity* of any network, defined as $\langle\sigma\rangle r/\hat{l}$, is bounded by the universal upper-bound $\frac{1}{3}$. Now we want to determine under what conditions its upper bound is attained. We shall show that the following two conditions are necessary and sufficient:

(a) all wires are straight;

(b) all junctions (i) between wires satisfy $\sum_j \boldsymbol{e}_{ij} = \boldsymbol{0}$, where $\boldsymbol{e}_{ij}$ are outward-pointing unit vectors in the directions of adjoining wires.

(*a*) *Necessity of conditions* (a) *and* (b)

We first prove the necessity of condition (a) as follows. Any infinitesimal change in the geometry of the network will imply a variation $\mathrm{d}\langle\sigma\rangle$ of its averaged conductivity and a variation $\mathrm{d}\hat{l}$ of its line length per unit volume. If the conductivity of the network was at its maximum value before the geometrical change, from equation (2.6) it follows that

$$\mathrm{d}\left(\frac{\langle\sigma\rangle}{\hat{l}}\right) = 0, \tag{3.1}$$

or, equivalently,

$$\mathrm{d}\langle\sigma\rangle = \frac{1}{3r}\,\mathrm{d}\hat{l}. \tag{3.2}$$

This equality has to be satisfied for any geometrical perturbation. Suppose the geometrical change was the increase in the length of one given wire, all the other wires and all the junctions staying unchanged. This can only increase the line length of the network, so $\mathrm{d}\hat{l} \geqslant 0$, and, from the monotonicity law, the averaged conductivity can only decrease (since we increase the length of one wire, the others being unchanged) and so $\mathrm{d}\langle\sigma\rangle \leqslant 0$. As a consequence, both the conductivity and the line length have to be stationary when the length of one wire is increased. We come to the same conclusion by decreasing the length of one given wire. The only way that any infinitesimal variation in the length of each wire does not change the line length of the network is that all the wires are straight, and so condition (a) is a necessary condition.

The necessity of condition (b) is less obvious. To demonstrate the existence of an upper bound of the conductivity in the x-direction, we used two successive inequalities. Firstly, the conductivity of the network is less than the conductivity of the same network intersected with zero resistance sheets. Secondly the conductivity of the network intersected with zero resistance sheets is itself bounded, using the fact that the equivalent resistance of N resistive elements in series arrangement is greater or equal to the equivalent resistance of the same resistive elements in parallel arrangement times N^2 (see Appendix B). So in order to get the exact upper bound, these two inequalities have to become strict equalities. The first one implies that the presence of sheets does not modify the distribution of potentials, and so the potential in the wires is a function of x only. To see this, increase progressively the resistance of the sheets up to infinity (corresponding to the initial network). From the monotonicity law, this can only decrease the conductivity of the network. The only way for the conductivity to stay at its maximum value while increasing the resistance of each sheet is that there is no current through the sheets, and so the potential in the wires is a function of x only.

The second equality requires that the resistances of every slice (of equal thickness) are the same. Indeed, the resistance of a slice of arbitrary thickness x is simply proportional to x, and it follows that the system is equivalent to a single *uniform* resistor. Hence the potential is indeed *linear* in x.

Examination of Kirchhoff's law (of charge conservation) at a vertex in a uniform field immediately leads to condition (b), or rather a single component of this vector equation:

$$\sum_j \cos\alpha_{ij} \cdot \operatorname{sgn}(x_i - x_j) = 0 \tag{3.3}$$

(the term $\operatorname{sgn}(x_i - x_j)$, which gives '$-1$', '0' or '1' depending on whether $(x_i - x_j)$ is negative, zero or positive, is introduced in order to satisfy $\cos\alpha_{ij} \geqslant 0$). The arguments holds in the two other directions, so condition (b) is indeed required.

(*b*) *Sufficiency of conditions* (a) *and* (b)

We now prove the sufficiency of these conditions. Suppose a network for which these conditions are satisfied, and upon which a potential difference U_x between its two corresponding faces is applied, in the x-direction. We define the trial potential function

$$\Psi = -\frac{U_x}{L_x} x$$

at all points of the network, where L_x is the distance between the two electrodes. We first check that Kirchhoff's laws are satisfied under the conditions stated. The first Kirchhoff law, which states that the sum of potential differences along a loop is null, is naturally satisfied. The second law, which states that the sum of algebraic currents in one junction is null, is also satisfied: the current in the straight wire (i, j) is given by

$$I_{ij} = -\frac{1}{r} \nabla\Psi \cdot \boldsymbol{e}_{ij} = \frac{U_x}{rL_x} \boldsymbol{e}_x \cdot \boldsymbol{e}_{ij}, \tag{3.4}$$

so

$$\sum_j I_{ij} = \frac{U_x}{rL_x} \boldsymbol{e}_x \cdot \sum_j \boldsymbol{e}_{ij} = 0. \tag{3.5}$$

The trial potential function also satisfies the boundary conditions, and so is the correct physical solution.

We now check that the conductivity is equal to the upper bound. The potential is uniform on planes parallel to the electrodes, and so the system is unaltered when intersected by thin parallel sheets of infinite conductivity, perpendicular to the x-direction. In Appendix B we calculate the elementary resistance $\Delta R(x)$ of a slice of thickness Δx at position x for such a network (equation (B 1)). The global current I_x, the elementary resistance $\Delta R(x)$ and the elementary potential difference $\Delta\phi$ across the slice are related by

$$\frac{\Delta\Psi}{\Delta x} = \frac{\Delta R}{\Delta x} I_x. \tag{3.6}$$

But here both the potential gradient and the global current are independent of x, and this also applies for $\Delta R/\Delta x$. Using equation (B 1) and integrating in the x-direction, we obtain

$$I_x \frac{L_x^2}{U_x} = \sum_{(i,j)} l_{ij} \frac{\cos^2 \alpha_{ij}}{r}. \tag{3.7}$$

From this expression we get the expression of the conductivity in the direction x:

$$\sigma_x = \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2 \alpha_{ij}}{r}. \tag{3.8}$$

This value of the conductivity corresponds to the upper bound of the conductivity along x, as expressed in equation (2.2). The same reasoning can be applied for the conductivity in the two other directions, and so the sufficiency of conditions (a) and (b) is proved.

(*c*) *Remarks*

We now comment on these necessary and sufficient conditions: it is remarkable that the maximum of the conductivity of the network is independent of the connectivity of the junctions: two networks with different topology but with the same line length per unit volume and which both satisfy conditions (a) and (b) have the same averaged conductivity (so the same conductivity if these networks are isotropic).

We have just established the two necessary and sufficient conditions on the geometry of a network in order to maximize the normalized conductivity $\langle\sigma\rangle r/\hat{l}$. It can also be shown that they are necessary and sufficient conditions in order to have a minimum of the line length per unit volume $\hat{l}$ for a given topology. Furthermore, if such a minimum exists, it is unique (Colthurst *et al.* 1993; Morgan 1994). So in a class of networks with the same topology, the normalized conductivity reaches the universal upper bound $\frac{1}{3}$ (or equivalently, the averaged conductivity reaches its relative maximum value $\hat{l}/3r$) when and only when the line length per unit volume is minimum. This result is not obvious and cannot simply be drawn from the monotonicity law: a small change of the position of a given junction from the minimal line-length configuration implies an increase in the total line length, but some wires reduce in length, whereas some others increase in theirs, and so the monotonicity law is unable to predict the variation of the network conductivity.

Moreover, since the two conditions are necessary and sufficient to maximize $\langle\sigma\rangle/\hat{l}$ and minimize $\hat{l}$ for a fixed topology, they are also sufficient to make $\langle\sigma\rangle$ maximum for this topology. However, it is not clear whether they are also necessary conditions (we do not know if there exist stationary points of $\langle\sigma\rangle$ that are not stationary points of $\langle\sigma\rangle/\hat{l}$ or $\hat{l}$, and we have to leave this point as an open question).

4. Relevance to foam conductivity

Foam is a dispersion of a gaseous phase in a liquid (or in a solid) phase. In the limit of *dry* foam, the liquid fraction, defined as the volume of liquid per unit of foam volume, tends to zero. In this case, bubbles have polyhedral shapes and we can distinguish three different geometrical elements in the foam structure: lamellae, channels (or Plateau borders), which are junction of lamellae, and nodes, which are junctions of channels. At the end of the nineteenth century the Belgian physicist J. A. F. Plateau studied the structure of such foams, and discovered the following geometrical rules for equilibrium (Plateau 1873).

(i) Only three lamellae are joined in one channel, the angle between two lamellae being 120°.

(ii) Only four channels are joined in one node in a tetrahedral configuration (the angle between two channels being $\arccos(-\frac{1}{3}) \simeq 109.5°$).

A third condition is imposed by the Laplace pressure/curvature relation for the films, but it does not concern us directly here. In the dry limit, the transverse curvature dominates and is equal throughout the network of channels. In consequence, the dry foam can be considered as an electrical network of interconnected wires, the resistance of each wire being proportional to the length of the corresponding Plateau border. The liquid fraction ϕ_{l} of the foam can also be related to its line length per unit volume by

$$\phi_{\mathrm{l}} = s\hat{l}, \tag{4.1}$$

where s is the cross-sectional area of Plateau borders. Using equation (1.3), it follows that the conductivity of the foam is bounded in the following way:

$$\frac{\sigma}{\sigma_{\mathrm{liq}}} \leqslant \tfrac{1}{3}\phi_{\mathrm{l}}. \tag{4.2}$$

Note that the necessary and sufficient condition (b) is always fulfilled in a foam in the dry limit, in which the Plateau borders become thin, due to the geometrical structure imposed by Plateau's laws. So the difference between the real value of the conductivity in the dry limit and the maximum value is due to the curvature of channels only. As the difference in pressure between adjacent bubbles is at the origin of curvature of the channels, we can expect that the conductivity of a quiet monodisperse foam is close to the upper bound value, which has been experimentally confirmed (Phelan *et al.* 1996). The curvature in question is inevitable in foams: Plateau's laws cannot be fully satisfied without them (Weaire & Hutzler 1999).

5. Examples

(*a*) *Straight-line networks*

We present various periodic networks made of straight wires and compare their conductivities with the upper bound. The basic mesh of the first network is the truncated cube. Its conductivity can be obtained analytically. The second and third networks are based on the Weaire–Phelan mesh and the Friauf–Laves mesh, respectively, and, due to the complexity of their structures, distributions of currents and potentials (Kirchhoff's laws) are found using Mathematica.

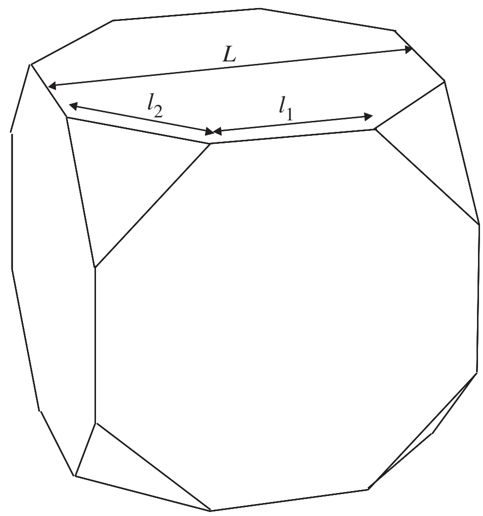

Figure 2. Structure of the truncated cube. The two different channel lengths l_1 and l_2 are related to the mesh length L by $l_1 + \sqrt{2}l_2 = L$.

(i) *Truncated cube*

The basic mesh of this isotropic network is depicted in figure 2.

The two different channel lengths l_1 and l_2 are related to the length of the mesh L by

$$L = l_1 + \sqrt{2}l_2. \tag{5.1}$$

We can easily obtain the conductivity of this network:

$$\sigma = \frac{1}{RL} = \frac{1}{(l_1 + \frac{1}{2}l_2)rL}. \tag{5.2}$$

Equation (5.2) can be rewritten in terms of the function of the line length,

$$\hat{l} = \frac{3l_1 + 12l_2}{L^3},$$

as

$$\sigma = \frac{L^2\hat{l}}{r(3l_1 + 12l_2)(l_1 + \frac{1}{2}l_2)}. \tag{5.3}$$

In figure 3, we plot

$$\frac{\sigma r}{L\hat{l}}$$

versus l_1/L. We can see that the function is always less than $\frac{1}{3}$, and equal to $\frac{1}{3}$ in the two limits $l_1 = 0$ and $l_1 = L$, which correspond to the cube-octahedral cell and the cubic cell, respectively. For these two cases condition (b) is fulfilled, so the Lemlich bound is attained.

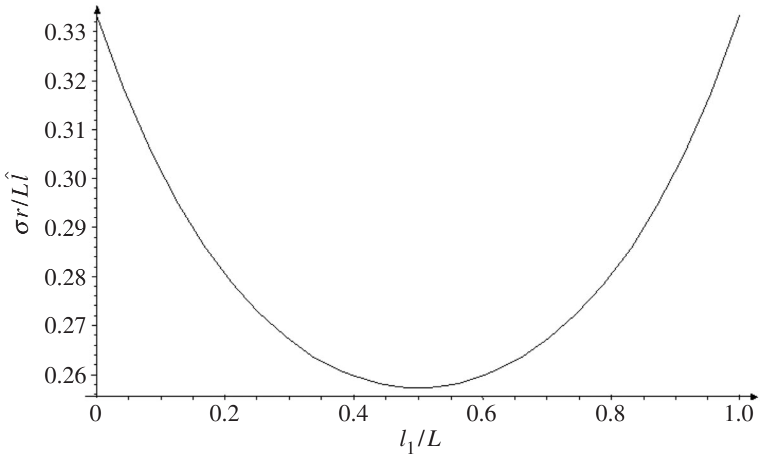

Figure 3. Graph of $\sigma r/L\hat{l}$ versus l_1/L for a network made of truncated cubes. The function is always less than $\frac{1}{3}$, and equal to $\frac{1}{3}$ in the two limits $l_1 = 0$ and $l_1 = L$, which correspond to the cubo-octahedral cell and the cubic cell, respectively. For these two cases condition (b) is fulfilled.

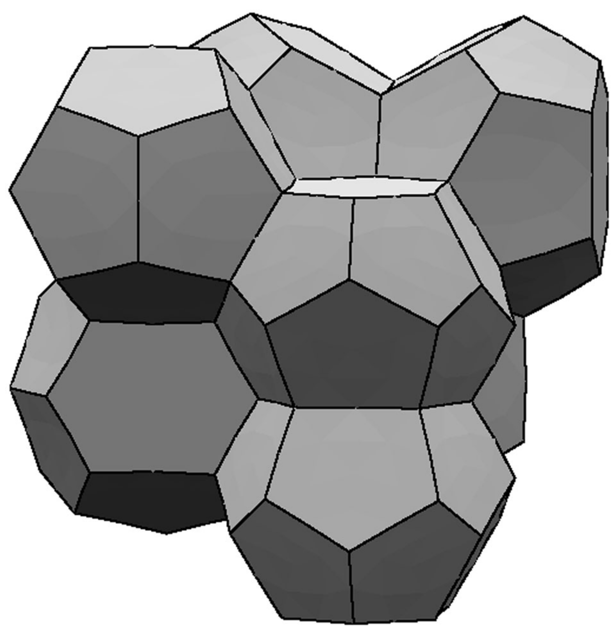

Figure 4. The Weaire–Phelan structure.

(ii) *Weaire–Phelan mesh*

The unit crystallographic mesh, depicted in figure 4, is composed of two polyhedra with 12 faces and six polyhedra with 14 faces (Weaire & Phelan 1994). The network is isotropic and the wires are straight and meet at a fourfold junction (as in a real foam) with angles which do not quite fulfil condition (b). Thus, we can expect that the conductivity of this network is only slightly less than the conductivity given by the Lemlich relation. Solving with MATHEMATICA, we find the following relation

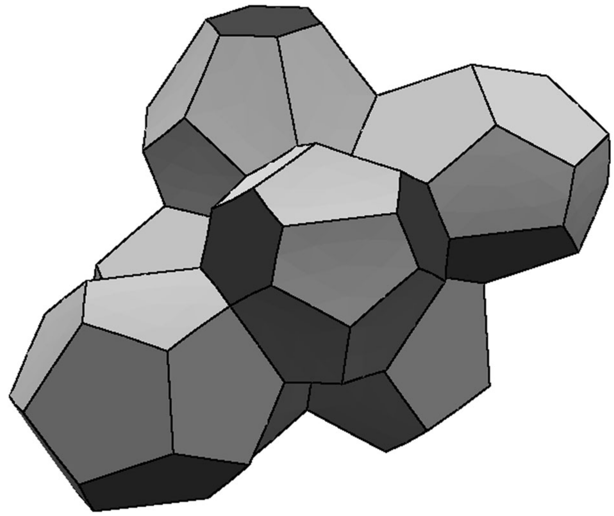

Figure 5. The Friauf–Laves structure.

between the effective conductivity and the line length per unit volume (Durand 2002):

$$\sigma = 0.331\frac{\hat{l}}{r}. \tag{5.4}$$

(iii) *Friauf–Laves mesh*

The unit crystallographic mesh is depicted in figure 5 and is composed of four polyhedra with 12 faces and two polyhedra with 16 faces. As for the previous structure, the network is isotropic and the wires are straight and meet at fourfold junction with angles slightly different from $\arccos(-\frac{1}{3})$. The relation obtained between the effective conductivity and the line length per unit volume is again (Durand 2002)

$$\sigma = 0.331\frac{\hat{l}}{r}. \tag{5.5}$$

(*b*) *The Kelvin cell and Kelvin foam structure*

It is also interesting to calculate the conductivity of the periodic network for which the unit mesh is the well-known Kelvin cell (or tetrakaidecahedron). This satisfies both conditions (a) and (b). Thus, the conductivity of this network is equal to the upper bound, as is easily checked by analytic solution of its conductivity. Although this structure does not satisfy Plateau's laws, it is still possible (as Lord Kelvin noticed (Thomson 1887)) to distort the faces (and edges) of the cells in such a way as to achieve this. Furthermore, the change of length is the same for all of the wires. The obtained structure is called 'Kelvin foam' (see figure 6). Let us call l_{s} and s_{s} the length and the cross-section of a straight wire, and l_{c}, s_{c} the length and the cross-section of a curved wire, respectively. The equivalent resistance of each kind of cell is proportional to the resistance of one wire of which the cell is composed, the constant of proportionality being the same, since the topology is unchanged. If R_{s} is the equivalent resistance of the cell made of straight wires, l_{s} the length of a

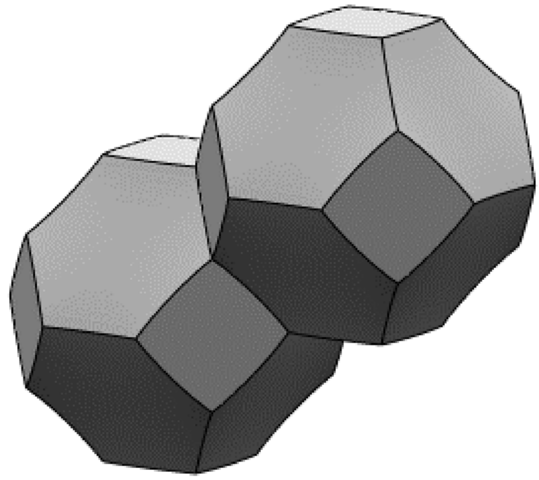

Figure 6. A Kelvin foam: the faces and the edges of each cell are curved in order to satisfy Plateau's laws.

straight wire, and s_s the cross-section of a straight wire, and R_c, l_c, s_c are the same quantities associated with the curved wires, then

$$\frac{R_\mathrm{s}}{R_\mathrm{c}} = \frac{l_\mathrm{s} s_\mathrm{c}}{l_\mathrm{c} s_\mathrm{s}}. \tag{5.6}$$

If the transformation from one structure to the other is done while keeping a constant liquid fraction, then $l_\mathrm{s} s_\mathrm{s} = l_\mathrm{c} s_\mathrm{c}$ and so the ratio of conductivities of the two structures is given by

$$\frac{\sigma_\mathrm{c}}{\sigma_\mathrm{s}} = \left(\frac{l_\mathrm{s}}{l_\mathrm{c}}\right)^2. \tag{5.7}$$

The ratio of conductivities thus varies as the square of the ratio of length wires. Numerical estimation of the ratio $l_\mathrm{s}/l_\mathrm{c}$ has been carried out by Princen & Levinson (1987): for cells of equal volume, they obtained $l_\mathrm{s}/l_\mathrm{c} = 1.154\,70/1.159\,64 \approx 0.995\,74$ and so the ratio of conductivities is

$$\frac{\sigma_\mathrm{c}}{\sigma_\mathrm{s}} \approx 0.991\,50. \tag{5.8}$$

6. Conclusions

We have established the existence of an upper bound for the averaged conductivity of a network made of resistive wires. The value of this upper bound is $\hat{l}/3r$, where $\hat{l}$ is the line length per unit volume of the network, and r is the resistance per unit wire length. We have also established the two necessary and sufficient conditions on the geometry of the network structure to reach the upper bound of conductivity: all the wires are straight and each junction (i) between wires satisfies $\sum_j \boldsymbol{e}_{ij} = \boldsymbol{0}$, where $\boldsymbol{e}_{ij}$ are outward-pointing unit vectors in the directions of adjoining wires. These two conditions also correspond to the necessary and sufficient conditions to have a minimum of the total line length of the network, for a fixed topology. It is perhaps surprising that the significance of the Lemlich law as an upper bound has escaped general notice up to this point (the use of the word 'limiting' in the title of Lemlich's paper refers only to the dry limit). However, while it is simply stated and understood, the general proof of law this does contain a number of technicalities, which are multiplied when the necessary and sufficient conditions for attainment of the bound are addressed, as here.

Knowledge of the relevant theorems is helpful in understanding why the Lemlich estimate works extremely well for realistic foam structures: they are very close to satisfying the conditions for exactness of the Lemlich estimate.

Real foams are often anisotropic (although this fact is commonly disregarded). This is because they have the distinctive property of accommodating very large strains in equilibrium, before reaching a yield stress. Under such strain, the structure, and hence the conductivity, becomes anisotropic. If we accept the Lemlich estimates of the components of anisotropic conductivity, this becomes closely related to stress in the case of a 2D foam, as conductivity and stress are related to the average of $\cos^2\theta$. A direct relation between these two physical quantities then follows, which should offer an interesting topic for a future experiment.

There is also scope for the extension of the results given here to related models, of relevance to the physics of foams. These include the case when a resistance is associated with each vertex, to represent the effects of finite liquid fraction (Phelan *et al.* 1996). Another model assigns conduction to the films, rather than the Plateau borders, and all three ingredients can be combined in an approximate but quite comprehensive model. These directions will be developed in future work.

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Appendix A. Averaged conductivity of a non-isotropic network

We define the averaged conductivity $\langle\sigma\rangle$ of a medium as the constant of proportionality between the averaged dissipated energy per unit volume and unit time $\langle P\rangle$ and the square of the external electric field modulus $|\boldsymbol{E}|^2$ when averaged on all the directions of the electric field (or equivalently of the medium):

$$\langle P\rangle = \langle\sigma\rangle|\boldsymbol{E}|^2. \tag{A 1}$$

The density of dissipative energy per unit time is

$$P = \boldsymbol{j}\cdot\boldsymbol{E} = \boldsymbol{E}\cdot[\sigma]\boldsymbol{E} = |\boldsymbol{E}|^2\boldsymbol{u}\cdot[\sigma]\boldsymbol{u}, \tag{A 2}$$

where $[\sigma]$ is the symmetrical conductivity tensor and $\boldsymbol{u}$ is the unit vector along the electric field. Then, by averaging,

$$\langle\sigma\rangle = \frac{1}{4\pi}\iint \boldsymbol{u}\cdot[\sigma]\boldsymbol{u}\,\mathrm{d}\omega = \frac{1}{4\pi}\sum_{i,j}\sigma_{ij}\iint u_i u_j\,\mathrm{d}\omega = \frac{1}{3}\sum_{i,j}\sigma_{ij}\delta_{ij}, \tag{A 3}$$

where $\mathrm{d}\omega$ is the elementary solid angle, σ_{ij} are the components of the conductivity tensor and u_i the components of $\boldsymbol{u}$. The averaged conductivity can be rewritten in terms of the function of the trace of the conductivity tensor:

$$\langle\sigma\rangle = \tfrac{1}{3}\,\mathrm{Tr}([\sigma]). \tag{A 4}$$

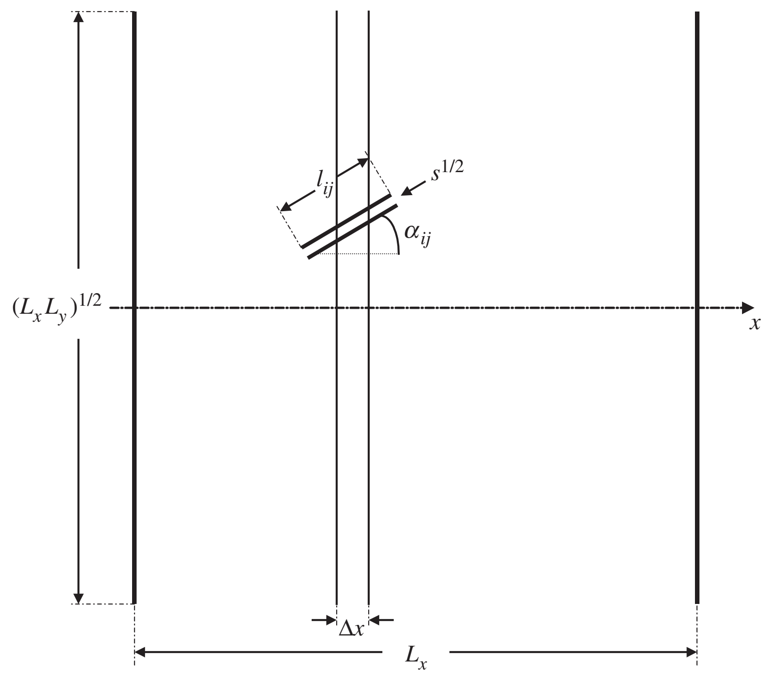

Figure 7. The contribution of the wire (i,j) to the conductance of a 'network slice', of thickness Δx, is $\cos\alpha_{ij}/(r\Delta x)$.

This result is not really surprising: $\langle\sigma\rangle$ can only be a function of the trace and the determinant of the tensor, which are the two invariants by changing the basis. But only the trace has the dimension of conductivity.

Appendix B. Upper bound of the conductivity of a network shorted with parallel sheets of infinite conductivity

We imagine a network made of straight wires with thin parallel sheets of infinite conductivity, perpendicular to the x-direction of the applied potential difference, and infinitesimally separated from each other by Δx. The potential is uniform on each sheet. The resistance $\Delta R(x)$ of the network slice at position x corresponds to the parallel association of the truncated resistive wires that it contains (we adjust the sheets in such a way that slices contain no junction). The resistance corresponding to the truncated channel (i,j) is equal to $r\Delta x/\cos\alpha_{ij}$, where r is the resistance per unit length of wire and α_{ij} is the angle between the channel (i,j) and the x-axis, as depicted in figure 7 (the angle is defined such that $\cos\alpha_{ij}\geqslant 0$):

$$\frac{1}{\Delta R(x)}=\sum_{(i,j)}P(x,x_i,x_j)\frac{\cos\alpha_{ij}}{r\Delta x}, \tag{B 1}$$

where $P(x,x_i,x_j)$ is a function which is unity if the channel (i,j) is intersected by the equipotential plane passing by x and zero otherwise:

$$P(x,x_i,x_j)=\begin{cases}1 & \text{if } \min(x_i,x_j)\leqslant x\leqslant\max(x_i,x_j),\\ 0 & \text{elsewhere.}\end{cases} \tag{B 2}$$

The total resistance is given by the sum of slice resistances. We call $N = L_x/\Delta x$ the number of slices (L_x is the length of the network in direction x on which the effective conductivity of the network is defined. For example, if the network is periodic, L_x is the spatial period along x). Thus,

$$R_x = \sum_{k=1}^{N} \Delta R\left(x = (k-1)\frac{L_x}{N}\right). \tag{B 3}$$

Now we need to use the following result.

If $\{f_1, f_2, \dots, f_N\}$ is a set of N real positive values, then

$$\langle f_k \rangle \langle f_k^{-1} \rangle \geqslant 1, \tag{B 4}$$

where

$$\langle f_k \rangle = \frac{1}{N}\sum_{k=1}^{N} f_k, \qquad \langle f_k^{-1} \rangle = \frac{1}{N}\sum_{k=1}^{N} \frac{1}{f_k}.$$

Moreover, it can be easily shown that the strict equality is attained when $f_1 = f_2 = \cdots = f_N$ only.

Using this theorem on the sum of equation (B 3), we obtain the inequality

$$\frac{1}{R_x} \leqslant \sum_{k=1}^{N} \sum_{(i,j)} P\left((k-1)\frac{L_x}{N}, x_i, x_j\right) \frac{\cos\alpha_{ij}}{r} \frac{\Delta x}{L_x^2}. \tag{B 5}$$

In the limit of indefinitely small Δx, the first sum in the right-hand term is substituted by an integral,

$$\frac{1}{R_x} \leqslant \int_0^{\infty} \sum_{(i,j)} P(x, x_i, x_j) \frac{\cos\alpha_{ij}}{r} \frac{\mathrm{d}x}{L_x^2}. \tag{B 6}$$

We can switch the sum and the integral of this expression, and it follows after integration that

$$\frac{1}{R_x} \leqslant \frac{1}{L_x^2} \sum_{(i,j)} l_{ij} \frac{\cos^2\alpha_{ij}}{r}. \tag{B 7}$$

The conductivity of the shorted network in the x-direction is defined by

$$\sigma_x^{(s)} = \frac{L_x}{L_y L_z R_x},$$

where L_y and L_z are network lengths in directions y and z. Thus,

$$\sigma_x^{(s)} \leqslant \frac{1}{V} \sum_{(i,j)} l_{ij} \frac{\cos^2\alpha_{ij}}{r}, \tag{B 8}$$

where $V = L_x L_y L_z$ is the volume occupied by the network.

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