Homework # 3 :

Interfaces and membranes : thermal fluctuations and Helfrich forces

Corrections

Ex I

1/ $g(\vec{r}) = \langle z(\vec{r})\, z(\vec{0}) \rangle$ $\qquad \vec{k} = (k_x, k_y)$

Expand in Fourier Series : $z(\vec{r}) = \sum_{\vec{k}} \tilde{z}(\vec{k})\, e^{i\vec{k}.\vec{r}}$ $\quad$ with $\tilde{z}(-\vec{k}) = \tilde{z}^*(\vec{k})$

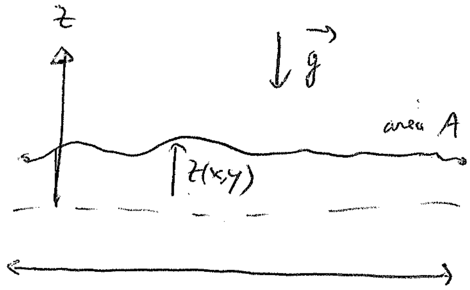
$z(\vec{0}) = \sum_{\vec{k}'} \tilde{z}(\vec{k}')$ $\quad$ (because $z(\vec{r}) \in \mathbb{R}$)

$$g(\vec{r}) = \sum_{\vec{k}} \sum_{\vec{k}'} \langle \tilde{z}(\vec{k})\, \tilde{z}(\vec{k}') \rangle e^{i\vec{k}.\vec{r}} = \sum_{\vec{k}} \sum_{\vec{k}'} \langle \tilde{z}(\vec{k})\, \tilde{z}(-\vec{k}') \rangle e^{i\vec{k}.\vec{r}}$$

$$= \sum_{\vec{k}} \sum_{\vec{k}'} \langle \tilde{z}(\vec{k})\, \tilde{z}^*(\vec{k}') \rangle e^{i\vec{k}.\vec{r}}$$

~~2/ $\langle \tilde{z}(\vec{k})\, \tilde{z}^*(\vec{k}') \rangle = \frac{k_B T}{\gamma L^2 (k^2 + k_c}$~~

2/ $$\mathcal{H} = \underbrace{\gamma A}_{\text{surface energy}} + \underbrace{\frac{\rho g}{2} \iint_{A_p} z^2(x,y)\, dx\, dy}_{\text{gravity energy}}$$



Projected area : $A_p = L \times L$

$$A = \iint_{A_p} \sqrt{1 + (\vec{\nabla} z)^2}\, dx\, dy \simeq A_p + \frac{1}{2} \iint_{A_p} (\vec{\nabla} z)^2\, dx\, dy$$

$$\Rightarrow \boxed{\mathcal{H} = \gamma A_p + \frac{\gamma}{2} \iint_{A_p} (\vec{\nabla} z)^2\, dx\, dy + \frac{\rho g}{2} \iint_{A_p} z^2(x,y)\, dx\, dy}$$

Fourier expansion : $z(\vec{r}) = \sum_{\vec{k}} \tilde{z}(\vec{k})\, e^{i\vec{k}.\vec{r}}$

$$\Rightarrow \boxed{\mathcal{H} = \frac{\gamma A_p}{2} \sum_{\vec{k}} \left(k^2 + \frac{\rho g}{\gamma}\right) |\tilde{z}(\vec{k})|^2} \quad + cst$$

$$\frac{\rho g}{\gamma} = \ell_c^{-2} \qquad \ell_c = \sqrt{\frac{\gamma}{\rho g}} : \text{ capillary length}$$

independent modes : (with Periodic B-C)

$$k_x = n_x \frac{2\pi}{L}, \quad \underline{\underline{n_x \in \mathbb{Z}}}$$

$$k_y = n_y \frac{2\pi}{L}, \quad \underline{\underline{n_y \in \mathbb{N}}}$$

$$\mathcal{H} = \gamma A_p \sum_{k_x=-\infty}^{+\infty} \sum_{k_y=0}^{+\infty} \left((\tilde{z}^R(\vec{k}))^2 + (\tilde{z}^I(\vec{k}))^2 \right)$$

$$\Rightarrow \langle \tilde{z}(\vec{k}) \tilde{z}^*(\vec{k}') \rangle = \frac{\int \prod_{k_x=-\infty}^{+\infty} \prod_{k_y=0}^{+\infty} d\tilde{z}^R(\vec{k})\, d\tilde{z}^I(\vec{k})\, e^{-\beta\gamma A_p \sum_{k_x, k_y \geq 0} (k^2+\ell_c^{-2})\left[(\tilde{z}^R_{\vec{k}})^2 + (\tilde{z}^I_{\vec{k}})^2\right]} \cdot \tilde{z}(\vec{k})\, \tilde{z}^*(\vec{k}')}{\int \prod_{k_x=-\infty}^{+\infty} \prod_{k_y=0}^{+\infty} d\tilde{z}^R_{\vec{k}}\, d\tilde{z}^I(\vec{k})\, e^{-\beta\gamma A_p \sum_{k_x, k_y \geq 0} (k^2+\ell_c^{-2})\left[(\tilde{z}^R_{\vec{k}})^2 + (\tilde{z}^I_{\vec{k}})^2\right]}}$$

$$= \begin{cases} 0 & \text{if } \vec{k}' \neq \vec{k} \\ \langle (\tilde{z}^R_{\vec{k}})^2 + (\tilde{z}^I_{\vec{k}})^2 \rangle = 2 \times \dfrac{k_B T/2}{\gamma A_p (k^2 + \ell_c^{-2})} & \text{if } \vec{k}' = \vec{k} \end{cases}$$

$$= \frac{k_B T}{\gamma L^2 (k^2 + \ell_c^{-2})}$$

3/ $\sum_{\vec{k}} \to \int d\vec{k}$ and $\delta_{\vec{k},\vec{k}'} \Rightarrow L\, \delta(\vec{k} - \vec{k}')$

4/ $$g(\vec{r}) = \sum_{\vec{k}} \frac{k_B T}{\gamma L^2 (k^2 + \ell_c^{-2})} e^{i\vec{k}.\vec{r}} = \iint \rho(\vec{k}) d\vec{k}\, \frac{k_B T}{\gamma L^2 (k^2 + \ell_c^{-2})} e^{i\vec{k}.\vec{r}}$$

$\rho(\vec{k})\, d^2\vec{k}$ = density of states $= \dfrac{dk_x\, dk_y}{\left(\frac{2\pi}{L}\right)\cdot\left(\frac{2\pi}{L}\right)} = \dfrac{A_p}{(2\pi)^2}\, dk_x\, dk_y$

Using polar coordinates:

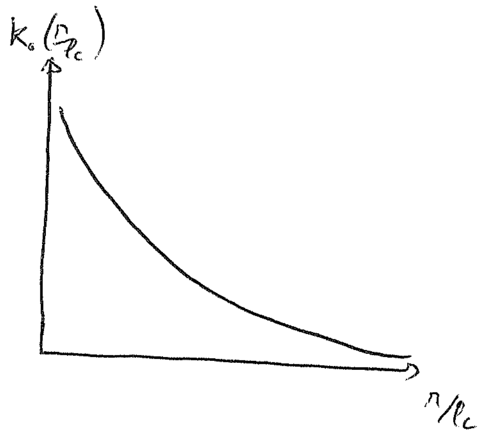
$$g(\vec{r}) = \frac{A_p}{(2\pi)^2} \iint k\,dk\,d\theta \; \frac{k_B T}{\underbrace{\sigma L^2}_{A_p}(k^2+\ell_c^{-2})} e^{ikr\cos\theta}$$

$$= \frac{k_B T}{(2\pi)^2 \gamma} \iint \frac{k\,dk\,d\theta}{k^2+\ell_c^{-2}} e^{ikr\cos\theta}$$

$$= \frac{k_B T}{\gamma (2\pi)^2} 2\pi \int_0^\infty \frac{k\,dk}{k^2+\ell_c^{-2}} J(0, kr)$$

$$\boxed{g(\vec{r}) = \frac{k_B T}{\gamma\, 2\pi} K_0\left(\frac{r}{\ell_c}\right)}$$

5/ when $r \gg \ell_c$: $K_0\left(\frac{r}{\ell_c}\right) \simeq \sqrt{\frac{\pi}{2}} \frac{e^{-r/\ell_c}}{\sqrt{r/\ell_c}}$



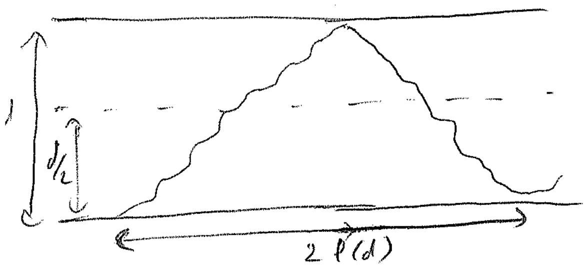
characteristic correlation length = $\underline{\underline{\ell_c}}$

Note: exponential decay: the interface positions are <u>short-range correlated</u>.

Ex 2

2.1 1/ $\ell(d)$ defined such that $\langle R^2(L=\ell(d))\rangle \simeq d$

$$\Leftrightarrow \frac{k_B T}{8\pi^3 \kappa_b}\,\ell(d)^2 = d^2$$



$$\Leftrightarrow \boxed{\ell(d) = d\sqrt{\frac{8\pi^3 \kappa_b}{k_B T}}}$$

2/ Pressure : $P = \frac{N k_B T}{V}$ where N = # of contact point on a wall

volume per contact point :

$$\frac{V}{N} = (2\,\ell(d))^2 \frac{d}{2} = 2\,\ell(d)^2\, d$$

$$\Rightarrow \boxed{P = \frac{k_B T}{2\,\ell(d)^2\, d}} \boxed{= \frac{(k_B T)^2}{16\,\pi^3 \kappa_b d^3}}$$

3/ $\Delta F = -\int_{-\infty}^{d} P(d')\,dd' = \frac{(k_B T)^2}{32\pi^3 \kappa_b d^2}$ $\Rightarrow$ algebraic decay

$\Rightarrow$ <u>long-range interactions</u>.

Note : $P(d) = -\frac{\partial F}{\partial d}$

4/ Van der Waals interaction : $\Delta F_{vdW} = -\frac{A}{d^2}$ where A indep of $T°$.

$\Rightarrow$ If $T \leq T_c$, membrane attracted by the wall.

If $T > T_c$, membrane can detach from the wall.

2.2

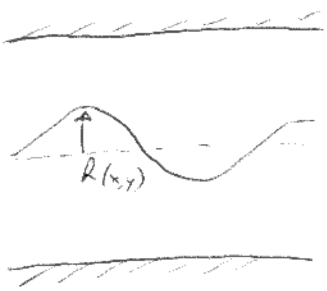
$$\Delta F = \Gamma k_B T$$

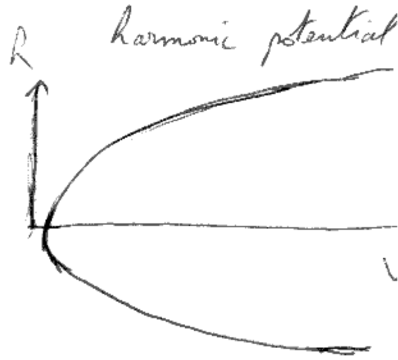
↳ # of contact points per unit surface area

$$\Gamma = 1/(2\,\ell(d))^2$$

$$\Rightarrow \boxed{\Delta F = \frac{(k_B T)^2}{32\pi^3 \kappa_b d^2}}$$

2.3 Quantitative approach





1/ $$\mathcal{H} = \frac{1}{2}\int dx\,dy\,\Big[\underbrace{\kappa_b(\nabla^2 h)^2}_{\text{bending energy}} + \underbrace{m h^2}_{\text{harmonic potential}}\Big]$$

$$= \frac{L^2}{2}\sum_{\{\vec{q}\}} (\kappa_b q^4 + m)\,|\tilde{h}(\vec{q})|^2$$

$$= L^2 \sum_{q_y>0}\sum_{q_x} (\kappa_b q^4 + m)\Big[(\tilde{h}^R_{\vec{q}})^2 + (\tilde{h}^I_{\vec{q}})^2\Big]$$

Equipartition theorem $\Rightarrow L^2(\kappa_b q^4 + m)\langle|\tilde{h}_{\vec{q}}|^2\rangle = \frac{k_B T}{2} + \frac{k_B T}{2}$

$$\Rightarrow \boxed{\langle|\tilde{h}_{\vec{q}}|^2\rangle = \frac{k_B T}{L^2(\kappa_b q^4 + m)}}$$

2/ $\langle h^2 \rangle = \sum_{\vec{q}} \langle |\tilde{h}_{\vec{q}}|^2 \rangle$ (Parseval theorem)

$$= \sum_{\vec{q}} \frac{k_B T}{L^2 (\kappa_b q^4 + m)}$$

$$= \int_0^\infty \rho(q)\,dq \cdot \frac{k_B T}{L^2(\kappa_b q^4 + m)}$$

where $\rho(q)$: density of state having $\vec{q} \in [q, q+dq[$

Periodic B.C. : $q_x = n_x \frac{2\pi}{L}$ $q_y = n_y \frac{2\pi}{L}$

$$\Rightarrow \rho(q)\,dq = \frac{2\pi q\,dq}{\left(\frac{2\pi}{L}\right)\left(\frac{2\pi}{L}\right)} = \frac{L^2}{2\pi} q\,dq$$

$$\Rightarrow \langle h^2 \rangle = \frac{k_B T}{2\pi} \int_0^\infty \frac{q\,dq}{\kappa_b q^4 + m}$$

$$\boxed{\langle h^2 \rangle = \frac{k_B T}{8\sqrt{\kappa_b m}}}$$

3/ m chosen such that $\sqrt{\langle h^2 \rangle} \simeq d$

$$\Leftrightarrow \frac{k_B T}{8\sqrt{\kappa_b m}} \simeq d^2$$

$$\Leftrightarrow \boxed{m \simeq \frac{(k_B T)^2}{\kappa_b d^4}}$$

4/ $Z = \int \mathcal{D}[h]\, e^{-\beta \frac{L^2}{2} \sum_{\{\vec{q}\}} (\kappa_b q^4 + m)|\tilde{h}_{\vec{q}}|^2}$

$$= \iiint_{-\infty}^{+\infty} \prod_{\substack{q_y \geq 0 \\ q_x}} d\tilde{h}^R_{\vec{q}}\, d\tilde{h}^I_{\vec{q}}\; e^{-\beta L^2 \sum_{\substack{q_y \geq 0 \\ q_x}} (\kappa_b q^4 + m)|\tilde{h}_{\vec{q}}|^2}$$

$$= \prod_{\substack{q_y \geq 0 \\ q_x}} \left(\sqrt{\frac{\pi}{\beta L^2 (\kappa_b q^4 + m)}}\right)^2$$

$$F = -k_B T \ln Z = -k_B T \sum_{\substack{q_y \geq 0 \\ q_x}} \ln\left(\frac{\pi}{\beta L^2 (\kappa_b q^4 + m)}\right) = -\frac{k_B T}{2} \int_0^{+\infty} \rho(q)\,dq\, \ln\left(\frac{\pi}{\beta L^2(\kappa_b q^4 + m)}\right)$$

$$\boxed{F = +\frac{k_B T}{4\pi} L^2 \int_0^{+\infty} q \ln\left(\frac{\beta L^2(\kappa_b q^4 + m)}{\pi}\right) dq}$$

5/ $$P(d) = -\frac{\partial (F/L^2)}{\partial d}$$

$$= -\frac{k_B T}{4\pi}\int_0^{+\infty} q \, \frac{\beta L^2 \frac{\partial m}{\partial d}}{\beta L^2 (\kappa_b q^4 + m)} \, dq$$

$$\frac{\partial m}{\partial d} = -\frac{4 (k_B T)^2}{\kappa_b d^5} = -\frac{4m}{d}$$

$$P(d) = \frac{k_B T}{\pi}\frac{m}{d} \underbrace{\int_0^{\infty} \frac{q\,dq}{(\kappa_b q^4 + m)}}_{= \frac{\pi}{4\sqrt{\kappa_b m}}} = \frac{k_B T}{4d}\sqrt{\frac{m}{\kappa_b}}$$

Using expression for m, it comes:

$$\boxed{P(d) = \frac{(k_B T)^2}{4 \kappa_b d^3}}$$