

Chapter 2

Surface energy :

macroscopic description

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• Bibliography for this lesson :

- PG De Gennes, F Brochard-Wyart, D Quéré, “Capillarity and wetting phenomena : drops, bubbles, pearls, waves” (english version)
- PG De Gennes, F Brochard-Wyart, D Quéré, “Gouttes, bulles, perles et ondes” (french version)
- L. Landau & E. Lifchitz, “Statistical Physics”

1 Surface tension

Here we briefly introduce the microscopic origin of surface energy between 2 immiscible phases (a more rigorous treatment will be carried in a next lesson).

1.1 Surface tension : an energy per unit surface

- Cohesive interactions in liquids and solids are responsible for surface tension :

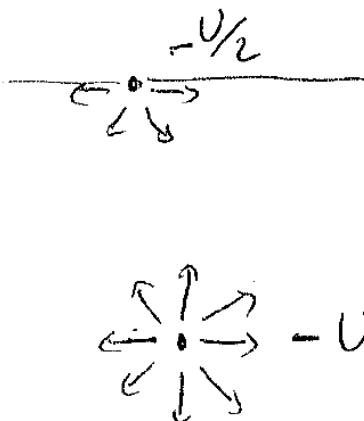


FIGURE 1 – Illustration with the interface between a fluid and the air : the cohesive energy per molecule in the bulk is $-U$. On surface, a molecule roughly loses half of its neighbours, so its energy is $\simeq -U/2$.

Therefore, **there is an energy cost associated with the creation of a surface.**

- The work δW required to bring δN molecules to the surface is proportional to δN .
- Assuming fixed surface concentration, the expansion of area δA is proportional to δN .
- Therefore, $\delta W \propto \delta A$. The constant of proportionality is called the **surface tension** γ :

$$\delta W = \gamma \delta A.$$

$[\gamma]$: energy/area.

- This is a qualitative description : flaw within the argument : why fixed surface concentration ?
- Order of magnitude :
 - $\gamma = \frac{|U|/2}{a}$ where a is the area per molecule. For alkanes $-(CH_2)_n-$, cohesive interactions : Van der Waals, so $|U| \sim k_B T \sim 4 \cdot 10^{-21} J$. For $a \simeq 8^2$, one has $\gamma \sim 25 \cdot 10^{-3} J$.
 - Surface tension is higher when cohesive energy is higher (e.g. water-air : $\gamma \sim 72 \cdot 10^{-3} J$).
- Value of γ depends on the 2 immiscible media in contact. e.g. : $\gamma_{water-air} \neq \gamma_{water-oil}$.
- γ between 2 immiscible media is necessarily positive.
- Dependence with temperature : γ decreases when T increases.
- γ depends on the presence or not of amphiphilic molecules (see next lecture).

1.2 Surface tension : a force per unit length

- Experiment : soap film “attached to” a wire frame : the mobile rod experiences a force that tends to decrease the surface area.
- Work to increase film of $d\mathbf{x}$:

$$\begin{aligned} dW &= -\mathbf{F} \cdot d\mathbf{x} = \gamma 2dA \quad (\text{because 2 air-liquid interfaces}) \\ &= \gamma 2Ldx \end{aligned}$$

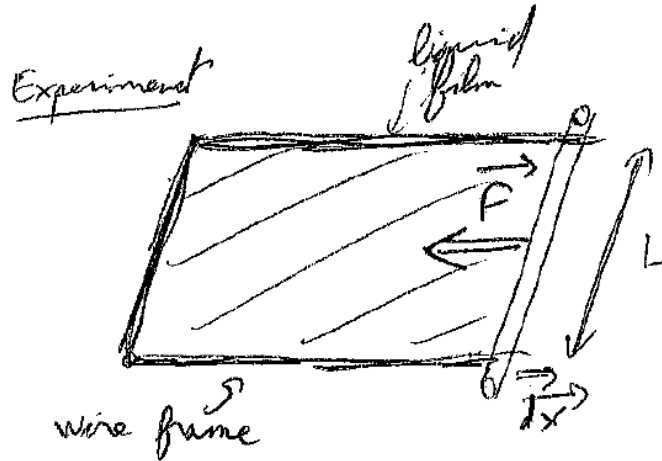


FIGURE 2 – Existence of a restoring force \mathbf{F} tangent to the liquid film.

Hence :

$$\mathbf{F} = -2\gamma L \mathbf{e}_x.$$

γ is a force per unit transverse length (and per interface).

- Units $[\gamma] = J/m^2 = N/m$.
- This restoring force differs from an elastic force : its amplitude is a constant (i.e., it does not depend on the surface increase), while an elastic force is proportional to the elastic elongation (the more you increase the surface, the more you increase the restoring force).
- Why the force is *tangent* to the interface ?
 - In a real air-fluid interface, the density of molecules decreases continuously to zero.
 - The mean intermolecular distance d_s on the sharp, idealized interface is less than in the bulk d_0 .
 - Thus, the molecules at the surface are under tension (each one experiences attractive forces from its neighbors).
 - This explains why the sliding rod experiences a tangential attractive force from the interface...

2 When surface effects are important ?

- Usual thermodynamic description of a fluid :
 - macrostate defined by state variables T, V, N only. Justification : contribution of surface area A negligible in the thermodynamic limit : when volume and number of molecules are scaled by a same factor λ , the surface area is scaled by a factor $\lambda^{2/3}$ only, if all dimensions of the system are scaled by a same factor $\lambda^{1/3}$.
 - However, for small systems, contribution of the surface energy to the thermodynamic description may not be negligible anymore. Complete state variables are T, V, N, A (or T, V_1, N_1, V_2, N_2, A if the system is composed of two immiscible fluids separated with an interface with area A).
 - Illustration : consider a tank with cross-sectional area A containing a volume V of fluid. Gravity energy : $E_p = \frac{\rho g}{2} H^2 A = \frac{\rho g}{2} V^2 / A$. Surface energy : $E_{surf} = \gamma A$. Thus, ratio :

$$\frac{E_{surf}}{E_p} = \left(\sqrt{\frac{\gamma}{\rho g}} \frac{A}{V} \right)^2 = \left(\ell_c \frac{A}{V} \right)^2$$

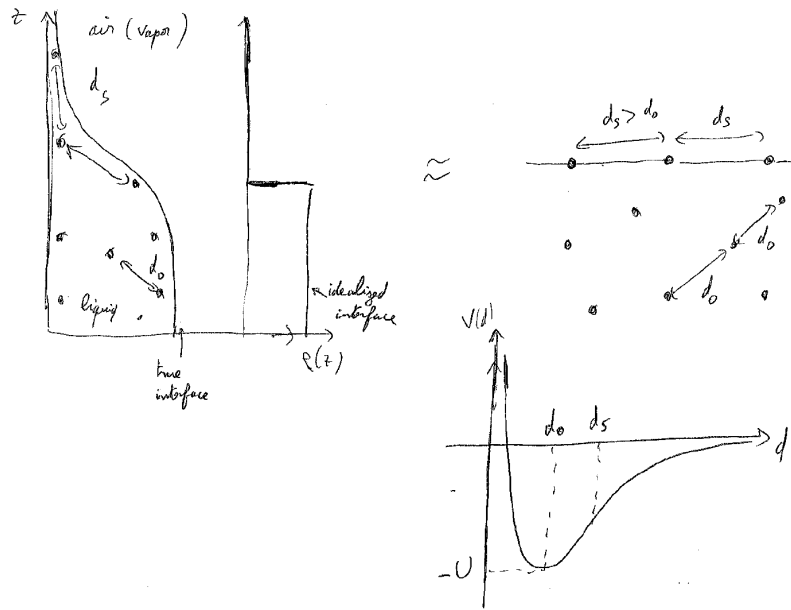


FIGURE 3 – Existence of a restoring force \mathbf{F} tangent to the liquid film.

$\ell_c = \sqrt{\gamma/\rho g}$ is the *capillary length*.

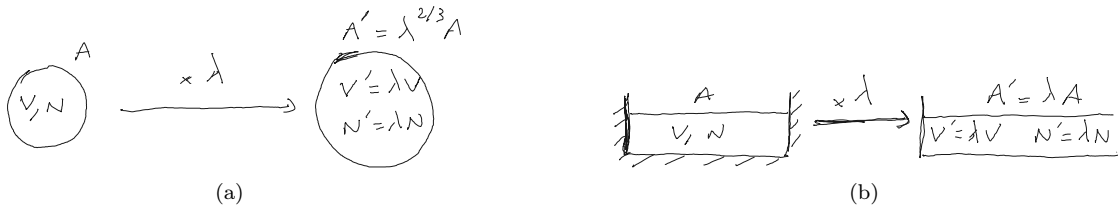


FIGURE 4 – (a) When all the dimensions are scaled by a same factor λ , the surface/volume ratio goes to 0 when $\lambda \rightarrow \infty$; (b) when you increase the lateral dimensions of a tank containing a fluid, the surface/volume ratio stays constant.

- Interfacial energy plays a non negligible role everytime the volume/surface ratio is $\ll \ell_c$:
 - spherical shape of a drop or a bubble
 - “walking” of insects on water surface
 - ascent of sap in plants
 - hair of a wet pencil stucked together
 - ...

3 Curvatures of a surface

Equilibrium configurations often correspond to non-flat interfaces : e.g. : shape of a small drop, meniscus of liquid near a wall,...

To study the equilibrium configuration of a surface, one needs to introduce first the surface curvatures.

3.1 Recap : curvature of a curve embedded in a 2D space

• definition :

- Consider a curve \mathcal{C} embedded in the euclidean 2D space.
- \mathbf{t}, \mathbf{n} : unit tangent and normal vectors at point M respectively.
- curvature κ of \mathcal{C} at point M is defined through :

$$\frac{d\mathbf{t}}{ds} = \kappa\mathbf{n},$$

where s is the curvilinear abscissa.

- $R = 1/\kappa$ is the radius of the osculating ('kissing') circle at M .
- note : $\mathbf{t} = \frac{d\mathbf{r}}{ds}$, where \mathbf{r} is the position of point M with respect to some referring frame. Hence :

$$\frac{d^2\mathbf{r}}{ds^2} = \kappa\mathbf{n}.$$

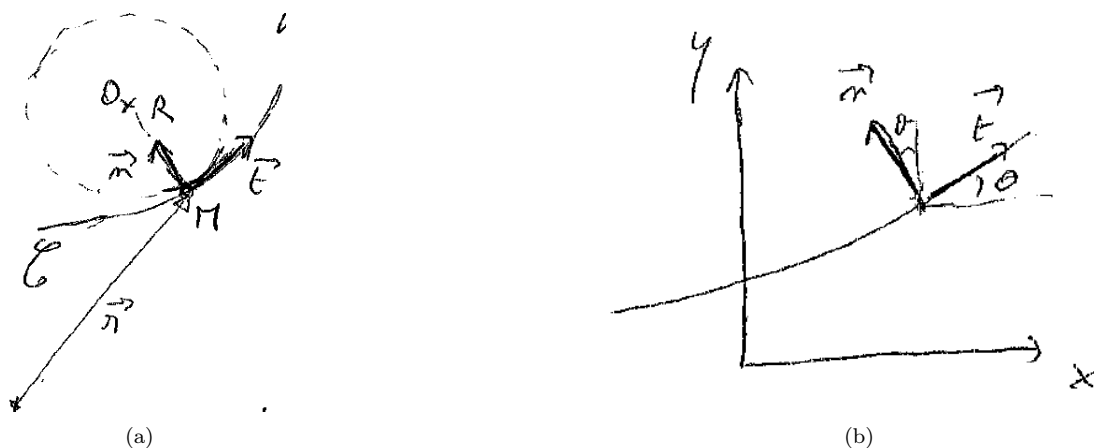


FIGURE 5 – (a) curvature of a curve in a 2D space ; (b) curvature in cartesian coordinates.

- **Gauss theorem (2D version)** : The integral of the curvature along a close, differentiable contour is a constant independent of the exact shape of this contour :

$$\oint \kappa ds = 2\pi n_t,$$

where n_t is the *turning number* of the curve. n_t increases from 1 for every counter-clockwise turn, and decreases from 1 for every clockwise turn. Examples :

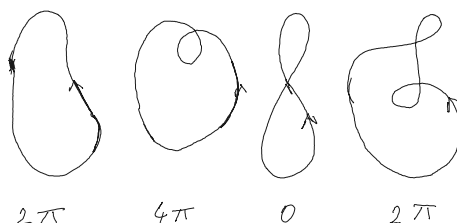


FIGURE 6 – Illustration of the 2D Gauss-Bonnet theorem for contours with different turning numbers.

- **Expression of κ in cartesian coordinates :**

- Let $y = f(x)$ be the equation of the curve \mathcal{C} in the Cartesian coordinate system Oxy .

- the tangent unit vector is $\mathbf{t} = \begin{vmatrix} \cos \theta \\ \sin \theta \end{vmatrix}$, and the normal unit vector $\mathbf{n} = \begin{vmatrix} -\sin \theta \\ \cos \theta \end{vmatrix}$.
- θ related to $f(x)$ through : $\tan \theta = df/dx = f'(x)$.
- Hence :

$$\mathbf{n} = \cos \theta \begin{vmatrix} -\tan \theta \\ 1 \end{vmatrix} = \frac{1}{\sqrt{1+f'^2(x)}} \begin{vmatrix} -f'(x) \\ 1 \end{vmatrix}.$$

- Moreover $ds = \sqrt{dx^2 + dy^2} = dx\sqrt{1+f'^2(x)}$. Therefore :

$$\frac{d\mathbf{t}}{ds} = \frac{1}{\sqrt{1+f'^2(x)}} \frac{d}{ds} \left(\begin{vmatrix} 1 \\ \sqrt{1+f'^2(x)} \end{vmatrix} f'(x) \right).$$

- Finally, comparing the two expressions, it comes :

$$\kappa = \frac{f''(x)}{(1+f'^2(x))^{3/2}}.$$

- note : the sign in this relation depends on the orientation of the y axis.
- for slight slopes (this is the case in particular near an extremum) : $|f'(x)| \ll 1 \Rightarrow \kappa \simeq f''(x)$.
- κ can also be related to 2D divergence of \mathbf{n} :

$$\kappa = \pm \operatorname{div} \mathbf{n}$$

(sign depends on the orientation of y axis and sign convention for curvature).

3.2 Curvatures of a surface embedded in a 3D space

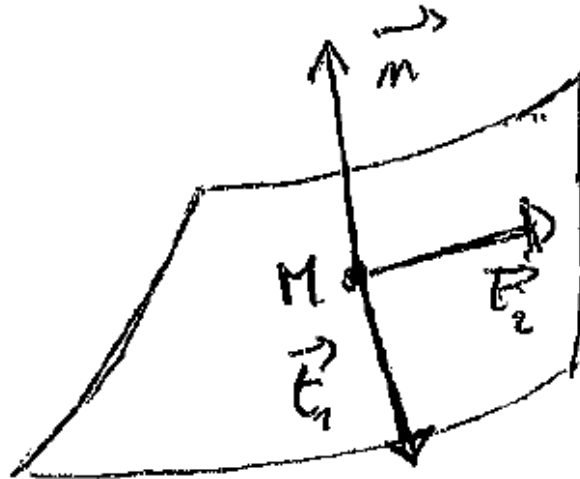


FIGURE 7 – Local cartesian frame at point M of surface \mathcal{S} : \mathbf{n} is the unit vector normal to the surface, and \mathbf{t}_1 and \mathbf{t}_2 are two unit tangent vectors at the surface which are orthogonal to each other.

- **Principal curvatures :**

- Following the definition used for a curve in the 2D space, one can define two different curvatures at a given point M of a differentiable surface \mathcal{S} :

$$\frac{d\mathbf{t}_1}{ds_1} = \kappa_1 \mathbf{n}, \quad \frac{d\mathbf{t}_2}{ds_2} = \kappa_2 \mathbf{n},$$

where s_1 and s_2 are the two curvilinear coordinates on \mathcal{S} .

- Values of κ_1 and κ_2 depend on the orientation of the tangent unit vectors $\mathbf{t}_1, \mathbf{t}_2$!
- Consider equation of \mathcal{S} in the fixed Cartesian frame $Mxyz$ with axes Mx, My, Mz that coincide with $\mathbf{t}_1, \mathbf{t}_2$ and \mathbf{n} at $M : z = f(x, y)$. Then :

$$\left(\frac{\partial f}{\partial x}\right)_M = \left(\frac{\partial f}{\partial y}\right)_M = 0.$$

- Let N be a point on \mathcal{S} close to M . The vertical distance between M and N is :

$$dz = \frac{1}{2} \sum_{i=1}^2 \sum_{j=1}^2 H_{ij} dx_i dx_j,$$

where $x_1 = x, x_2 = y$, and $H_{ij} = \left(\frac{\partial^2 f}{\partial x_i \partial x_j}\right)_M$. H_{ij} are the components of the Hessian matrix \mathbf{H} . Note that the diagonal elements of \mathbf{H} are the curvatures κ_1 and κ_2 .

- \mathbf{H} is symmetric, and hence diagonalizable. That means there is a specific orientation for the tangent vectors \mathbf{t}_1 and \mathbf{t}_2 for which \mathbf{H} is diagonal. This specific reference frame is called **principal axes** and the eigenvalues are the **principal curvatures** κ_1^0 and κ_2^0 .
- It can be shown that the two principal curvatures correspond to the minimal and maximal values of curvatures that can be measured by rotating the tangent vectors $\mathbf{t}_1, \mathbf{t}_2$.

– Examples :

Plane : $\kappa_1^0 = \kappa_2^0 = 0$.

Sphere with radius R : $\kappa_1^0 = \kappa_2^0 = 1/R$.

Cylinder with radius R : $\kappa_1^0 = 1/R$ and $\kappa_2^0 = 0$.

Catenoid : $\kappa_1^0 = -\kappa_2^0$.

• **Gaussian and mean curvatures :**

- Any scalar local quantity which depends on the curvatures of the surface (e.g. : energy) must be independent of the choice of the frame in which they are calculated. Therefore, its expression must be a combination of the two invariants of the Hessian matrix : its trace and determinant.
- One defines the *mean curvature* as $H = \text{Tr } \mathbf{H} = \kappa_1 + \kappa_2 = \kappa_1^0 + \kappa_2^0$.
- One defines the *Gaussian curvature* as $G = \text{Det } \mathbf{H} = H_{11}H_{22} - H_{12}^2 = \kappa_1^0 \kappa_2^0$.

• **Gauss theorem (3D version) :** The integral of the Gaussian curvature over a closed, differentiable surface is a constant independent of the exact shape of this surface :

$$\boxed{\iint G d^2 S = 4\pi(1 - g),}$$

where g is the number of “holes” of the surface, e.g. : $g = 0$ for a sphere, $g = 1$ for a torus.

• **Expression of H in Cartesian coordinates :**

- As for the 2D case, one has : $H = \text{div } \mathbf{n}$.
- Let $z = f(x, y)$ be the equation of the surface :

$$\mathbf{t}_1 \propto \begin{vmatrix} 1 \\ 0 \\ \frac{\partial f}{\partial x} \end{vmatrix} \quad \mathbf{t}_2 \propto \begin{vmatrix} 0 \\ 1 \\ \frac{\partial f}{\partial y} \end{vmatrix}$$

and

$$\mathbf{n} = \mathbf{t}_1 \times \mathbf{t}_2 = \frac{1}{\sqrt{1 + f_x^2 + f_y^2}} \begin{vmatrix} -f_x \\ -f_y \\ 1 \end{vmatrix}$$

using the notation $f_x = \frac{\partial f}{\partial x} \dots$

– It comes :

$$H = \frac{(1 + f_x^2)f_{yy} + (1 + f_y^2)f_{xx} - 2f_x f_y f_{xy}}{(1 + f_x^2 + f_y^2)^{3/2}}$$

• Expression of G in Cartesian coordinates :

$$G = \frac{f_{xx}f_{yy} - f_{xy}^2}{(1 + f_x^2 + f_y^2)^2}$$

4 Young-Laplace law (1805)

• This law relates (locally) the shape of the interface between two immiscible fluids to the forces acting on it.

– For the moment, we assume that both fluids are **at rest**.

– We calculate the balance of forces acting on a control volume overlapping a small portion of the interface of surface area d^2S . Let dz_{in} and dz_{out} the (infinitesimal) thickness of the control volume in the “inside” and “outside” fluids, respectively.

– The force acting on the control volume are :

- Weight : $\mathbf{P} = (\rho_{in}dz_{in} + \rho_{out}dz_{out})d^2Sg\mathbf{e}_z$, where ρ_i are the respective densities of the fluids, and \mathbf{e}_z is the vertical unit vector.
- Hydrostatic pressure acting on the 6 faces of the control volume : $\mathbf{F}_p = (P_{out} - P_{in})d^2S\mathbf{n} + \mathcal{O}(dz_{in}, dz_{out})$, where \mathbf{n} is the normal unit vector pointing towards the inside fluid. $\mathcal{O}(dz_{in}, dz_{out})$ terms corresponds to the 4 faces that cross the interface.
- surface tension : $\mathbf{F}_s = \gamma dl_1(\mathbf{t}_1 + \mathbf{t}'_1) + \gamma dl_2(\mathbf{t}_2 + \mathbf{t}'_2)$ where the \mathbf{t}_i and \mathbf{t}'_i are the tangent unit vectors on the 4 sides of the small portion of interface.

From symmetry, one has : $(\mathbf{t}_1 + \mathbf{t}'_1) = 2 \sin(d\theta_2/2)\mathbf{n} = (d\theta_2 + \mathcal{O}(d\theta_2^3))\mathbf{n}$.

Similarly, $(\mathbf{t}_2 + \mathbf{t}'_2) = (d\theta_1 + \mathcal{O}(d\theta_1^3))\mathbf{n}$. Noticing that $d\theta_i = dl_i/R_i$ (where R_i are the radii of curvature) and $d^2S = dl_1dl_2$, the balance of forces yields, in the limit $dz_{in}, dz_{out} \rightarrow 0$, the

Young-Laplace law :

$$P_{in} - P_{out} = \gamma \left(\frac{1}{R_1} + \frac{1}{R_2} \right) = \gamma H.$$

Note : this relation still holds when the two centers of curvature are not on the same side of the interface, using the following sign convention for the two radii of curvature : $R_i > 0$ if the associated center of curvature is in the inside fluid zone, and $R_i < 0$ if it is in the outside fluid zone.

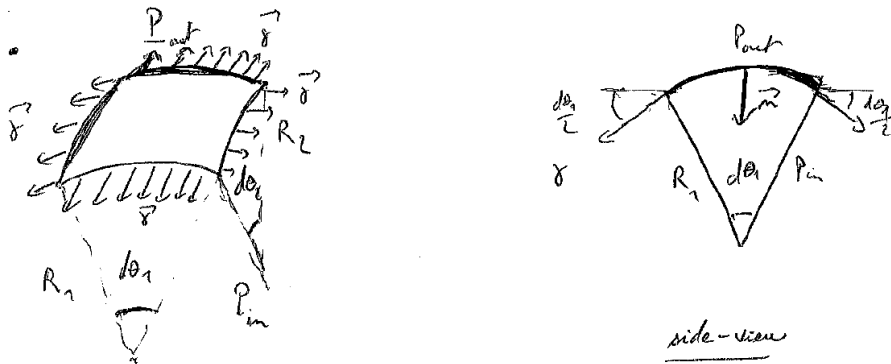


FIGURE 8 – Derivation of Young-Laplace law based on force balance arguments.

- **Exercise** : alternative demonstration, based on energy argument :
 - The equilibrium configuration corresponds to a minimum of (free) energy of the system, so any infinitesimal perturbation of the surface should not change the energy of the system.
 - Suppose each point M of the interface undergoes infinitesimal displacement $\delta l(M)\mathbf{n}$.
 - The elementary surface area $d^2S = R_1 d\theta_1 R_2 d\theta_2$ surrounding point M becomes :

$$d^2S' = (R_1 + \delta l)d\theta_1(R_2 + \delta l)d\theta_2 = d^2S(1 + H\delta l) + \mathcal{O}\left(\left(\frac{\delta l}{R_1}\right)^2, \left(\frac{\delta l}{R_2}\right)^2\right),$$

and so the change of area is $\delta(d^2S) = H\delta l d^2S$.

- The associated change of surface energy is :

$$\delta E_s = \gamma \iint H\delta l d^2S.$$

- The change of compressive energy of the inner and outer fluids is

$$\delta E_{comp} = \iint \mathbf{F} \cdot \boldsymbol{\delta} l d^2S = \iint P_{out} \delta l d^2S - \iint P_{in} \delta l d^2S.$$

Since one must have $\delta E = 0$ for any perturbation field δl , it comes :

$$\boxed{P_{in} - P_{out} = \gamma H.}$$

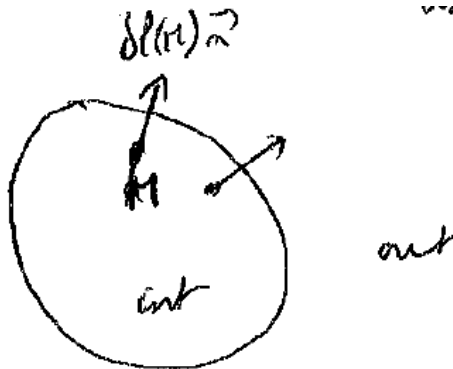


FIGURE 9 – Local cartesian frame at point M of surface S : \mathbf{n} is the unit vector normal to the surface, and \mathbf{t}_1 and \mathbf{t}_2 are two unit tangent vectors at the surface which are orthogonal to each other.

- Application of Laplace law to simple geometries :
 - spherical drop with radius R : $P_{in} - P_{out} = 2\gamma/R$.
 - a bubble in a liquid (inverted drop) with radius R : $P_{in} - P_{out} = 2\gamma/R$.
 - a bubble in the air (2 air-liquid interfaces) with radius R : $P_{in} - P_{out} = 4\gamma/R$.
 - a cylindrical interface, with radius R : $P_{in} - P_{out} = \gamma/R$.
 - a catenoid : $P_{in} - P_{out} = 0$.
- Generalization to the interface between fluids **in motion** :
 - let's go back to the argumentation based on forces. The two differences with the static case are :
 - the sum of forces is now equal to the inertia of the control volume. However, this inertia is proportional to the volume, hence $\mathcal{O}(dz_{in}, dz_{out})$. Therefore it will be negligible in the limit $dz_{in}, dz_{out} \rightarrow 0$.

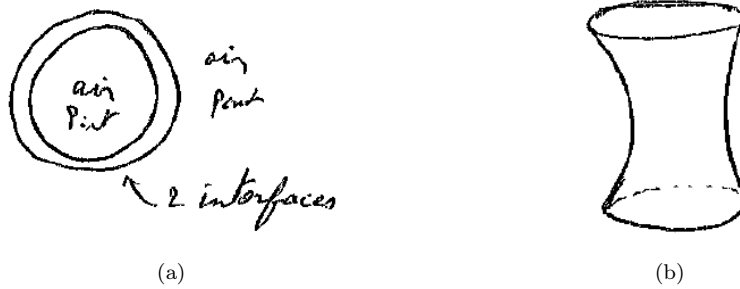


FIGURE 10 – (a) A bubble in the air has two close air-liquid interfaces; (b) for a catenoid, the pressure is the same on the two sides of the interface.

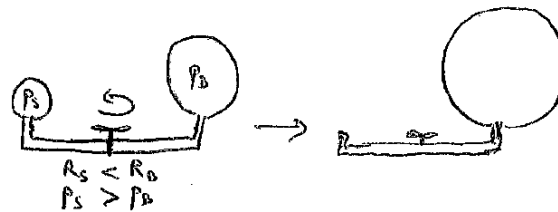


FIGURE 11 – Two bubbles — a small one and a large one — are connected to each other. According to Laplace law, the pressure is higher in the smaller bubble. Hence, the smaller bubble will empty in the larger one when they are connected.

- the forces generated by the fluids are not caused by the hydrostatic pressure solely : viscous stress must also be considered. Introducing the stress tensors σ_{in} and σ_{out} associated with each fluids, the force exerted by the fluids over the control volume is :

$$\mathbf{F}_p = \mathbf{n} \cdot ((\sigma_{in} - \sigma_{out}) \cdot \mathbf{n} d^2 S) + \mathcal{O}(dz_{in}, dz_{out})$$

(again, the $\mathcal{O}(dz_{in}, dz_{out})$ terms come from the force exerted on the 4 faces of the volume control that cross the interface). Note that by symmetry, the leading term is still normal to the surface element.

Hence the Young-Laplace law generalizes as follows for the interface between fluids in motion :

$$\boxed{\mathbf{n} \cdot ((\sigma_{in} - \sigma_{out}) \cdot \mathbf{n}) = \gamma H.}$$

Hint : for an incompressible, newtonian fluids, the stress tensor is : $\sigma_{ij} = -P\delta_{ij} + \eta \left(\frac{\partial v_i}{\partial x_j} + \frac{\partial v_j}{\partial x_i} \right)$, where η denotes the shear viscosity.

5 Wetting phenomena

- We consider systems with 3 immiscible media in contact, 1 solid and 2 fluids ; *e.g.*: solid (S), liquid (L) and gas (G). Thus, 3 surface tensions γ_{SG} , γ_{SL} , and γ_{LG} come into play.
 - If $\gamma_{SL} + \gamma_{LG} < \gamma_{SG}$, liquid spreads totally : *total wetting*.
 - If $\gamma_{SL} + \gamma_{LG} > \gamma_{SG}$, equilibrium configuration corresponds to a compromise : *partial wetting*.
- We call **contact line** the common frontier to the 3 media. We call **contact angle** θ_c the angle between the LG and LS interface at the contact line.

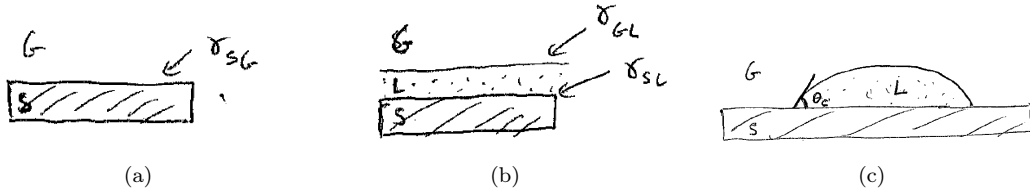


FIGURE 12 – (a) Solid-Gas interface ; (b) total wetting ; (c) partial wetting.

– If $0 < \theta_c < \pi/2$, the substrate is said *hydrophilic* (assuming liquid is aqueous solution). Hydrophilic surfaces are solids with high cohesive interactions (ionic or covalent bonds), *e.g.*: metal, glass, ionic crystal.

$$\gamma_{SG} \sim |U|/(2a^2) \sim 1\text{eV}/20\text{nm}^2 \sim 1\text{N/m}.$$

– If $\pi/2 < \theta_c < \pi$, the substrate is said *hydrophobic*. Hydrophobic surfaces are solids with low cohesive interactions (Van der Waals), *e.g.*: plastic materials

$$\gamma_{SG} \sim |U|/(2a^2) \sim 1k_B T/20\text{nm}^2 \sim 10\text{mN/m}.$$

5.1 Dupré-Young law

• To solve the exact shape of the interface, we need a boundary condition, *e.g.*: consider a drop of given volume. in absence of gravity, Young-Laplace law says that the drop surface is a spherical cap. Yet, it can intercept the substrate with many possible values of contact angle, while still satisfying the volume constraint.

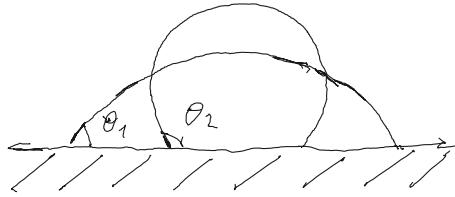


FIGURE 13 – What is the contact angle that gives the correct drop shape?

• Dupré-Young law reveals that θ_c depends on γ_{SG} , γ_{SL} , and γ_{LG} solely. Its value is not imposed by the geometry : instead, its value is fixed by the different affinities between the 3 media in contact, and will affect the geometry of the LG interface in return.

• **Exercise** : demonstration based on energy argument :

– We consider a very large drop (so we can forget the constraint on fixed volume). At equilibrium, a displacement δx of the contact line should not change the energy (per unit length) at first order.

– $\delta E_s = (\gamma_{SL} - \gamma_{SG})\delta x + \gamma_{LG}\delta l$, with $\delta l = \cos \theta_c \delta x$.

– Therefore,

$$\delta E_s = 0 \Leftrightarrow \boxed{\cos \theta_c = \frac{\gamma_{SG} - \gamma_{SL}}{\gamma_{LG}}}.$$

• Demonstration based on force balance argument :

– Consider the small control volume containing the contact line. The Forces acting on this volume dL^3 are :

◦ weight $\propto dL^3$

◦ surface forces due to hydrostatic pressure $\propto dL^2$

◦ line forces due to surface tensions $\gamma_{SG} + \gamma_{SL} + \gamma_{LG} \propto dL^0$

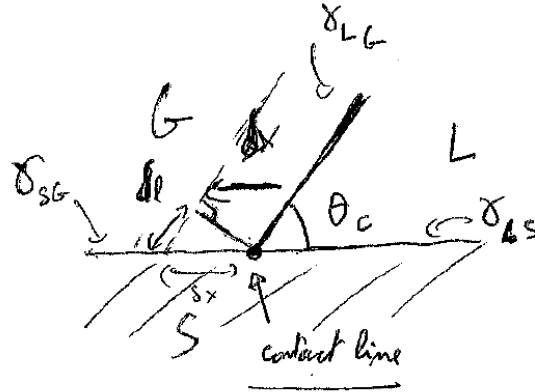


FIGURE 14 – Close-up of the contact line displaced from δx .

◦ reaction of the solid substrate $\mathbf{R} \propto dL^0$

In the limit $dl \rightarrow 0$:

- vertical balance does not help, because \mathbf{R} is unknown.
- horizontal balance leads to $\gamma_{SG} = \gamma_{SL} + \gamma_{LG} \cos \theta_c$, that is, to Dupré-Young law.

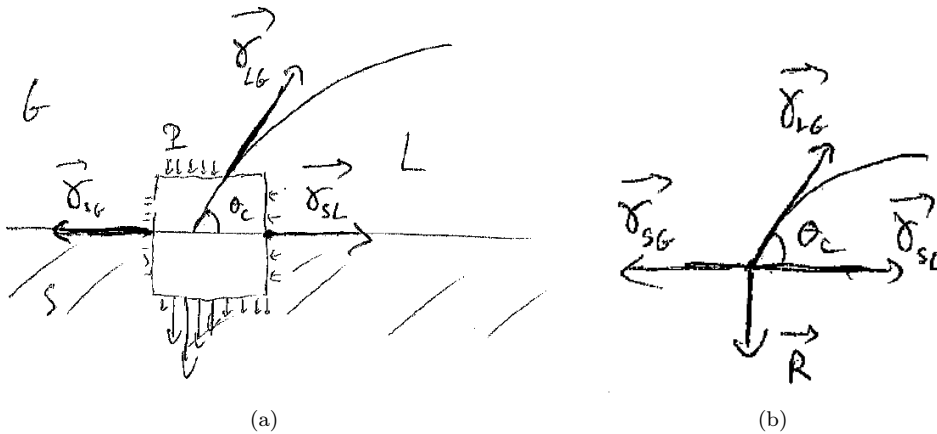


FIGURE 15 – (a) Force balance on a control volume that includes the contact line; (b) in the limit $\delta L^3 \rightarrow 0$, surface tensions and normal reaction of the substrate must balance.

5.2 Meniscus

- We want to study the equilibrium shape of the LG interface near a vertical wall.
 - Note : we look at length scale \gg amplitude of thermal fluctuations, so we consider that the interface is smooth (we will see later that this implies a renormalization surface tension). We note p_0 the (constant) atmospheric pressure, ρ the density of the liquid, γ the air-liquid surface tension, and $z_s(x)$ the equation of the interface (that we try to determinate).
 - Hydrostatic : $\rho \mathbf{g} - \nabla p = \mathbf{0} \Leftrightarrow p(z) = -\rho g z + \text{cst}$.
 - Laplace's law : $p_0 - p(z = z_s(x)) = \gamma \kappa(z_s(x)) = \gamma z_s'' / (1 + z_s'^2)^{3/2}$.
 - Far from the wall ($x \rightarrow \infty$) : $z_s \rightarrow 0$ and $\kappa \rightarrow 0$. Thus, $p = p_0 - \rho g z$.
 - Equation of surface :

$$z_s(x) = \ell_c^2 \frac{z_s''}{(1 + z_s'^2)^{3/2}},$$

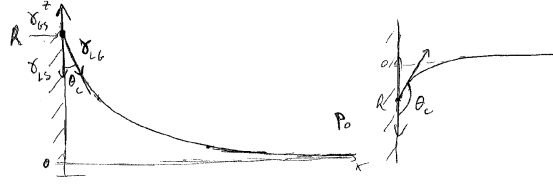


FIGURE 16 – Meniscus at the vicinity of a vertical wall. If $\theta_c < \pi/2$, the liquid climbs on the surface. If $\theta_c > \pi/2$, the meniscus is oriented downwards.

where $\ell_c = \sqrt{\gamma/\rho g}$ is the *capillary length*.

– Multiplying this equation by z'_s and integrating once, it comes :

$$z_s^2(x) = 2\ell_c^2 \left(C - \frac{1}{\sqrt{1 + z_s'^2}} \right).$$

– When $x \rightarrow \infty$, $z_s \rightarrow 0$, $z'_s \rightarrow 0$. Thus $C = 1$:

$$z_s^2(x) = 2\ell_c^2 \left(1 - \frac{1}{\sqrt{1 + z_s'^2}} \right).$$

– From this equation, typical height of the meniscus : $h \simeq \ell_c$. More rigorously : at $x = 0$, $z_s = h$ and $z'_s = -1/\tan\theta_c$, thus :

$$h = \sqrt{2}\ell_c \sqrt{1 - \sin\theta_c}.$$

– Typical width? When $|z'_s| \ll 1$, equation of surface : $z_s \simeq \ell_c^2 z_s''$, hence $z_s \sim e^{-x/\ell_c}$. Typical width $\simeq \ell_c$.

– Exact shape of the interface (implicitly) given by :

$$x - x_0 = \ell_c \operatorname{arsinh} \left(\frac{2\ell_c}{z_s} \right) - 2\ell_c \sqrt{1 - \left(\frac{z_s}{2\ell_c} \right)^2}.$$